LECTURE 12: LENGTH AND AREA.

October 31, 2001

Summary: Working in a parameterization. We can make calculations involving the *intrinsic geometry* of a surface by working in a parameterization.

	PARAMETERS		SURFACE
POINTS:	(u,v)	$\leftrightarrow$	$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.
VECTORS:	(a,b)	$\leftrightarrow$	$d\mathbf{r}_{(u,v)}(a,b) = a\dfrac{\partial \mathbf{r}}{\partial u}(u,v) + b\dfrac{\partial \mathbf{r}}{\partial v}(u,v)$.

FIRST FUNDAMENTAL FORM:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle \\ \left\langle \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle \end{pmatrix} \quad \leftrightarrow \quad \langle\, ,\, \rangle$$

$$(a_1, b_1)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad \leftrightarrow \quad \langle d\mathbf{r}(a_1,b_1), d\mathbf{r}(a_2,b_2)\rangle$$

$$Edu^2 + 2Fdudv + Gdv^2 \quad \leftrightarrow \quad ds^2$$

$$\sqrt{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \quad \leftrightarrow \quad dA$$

Arclength. Let $\alpha : (0,T) \to S$ be a smooth curve in the regular surface S. The length of α is

$$\int_0^T |\alpha'(t)|\, dt = \int_0^T \langle \alpha'(t), \alpha'(t)\rangle^{1/2}\, dt.$$

Because $\alpha'(t)$ lies in $T_{\alpha(t)}S$, we can compute this using the first fundamental form.
Now suppose that $\mathbf{r} : U \to S$ is a parameterization of S and $\beta : (0,T) \to U$ is a smooth curve, $\beta(t) = (u(t), v(t))$. Set $\alpha = \mathbf{r} \circ \beta$. Then the length of α is

$$\int_0^T \sqrt{E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(t))^2}\, dt.$$

Example. On the cylinder parameterized by

$$\mathbf{r}(\theta, z) = (\cos\theta, \sin\theta, z), \qquad 0 < \theta < 2\pi.$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Write down an integral equal to the length of the curve

$$\alpha(t) \;=\; \mathbf{r}(t,t) \;=\; (\cos t, \sin t, t), \qquad\qquad 0 < t < \pi.$$

Solution. Using the parameters, $(u(t), v(t)) = (t, t)$ so $u' = v' = 1$. The length is

$$\int_0^\pi \sqrt{E + 2F + G}\, dt = \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi.$$

Example. On the sphere parameterized by

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \qquad\qquad 0 < \phi < \pi, \quad 0 < \theta < 2\pi,$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}.$$

Write down an integral giving the length of the curve

$$\alpha(t) \;=\; \mathbf{r}(t,t) \;=\; (\sin t\cos t, \sin t\sin t, \cos t), \qquad\qquad 0 < t < \pi.$$

Solution. $(u(t), v(t)) = (t, t)$, $u' = v' = 1$. The length is

$$\int_0^\pi \sqrt{E + 2F + G}\, dt \;=\; \int_0^\pi \sqrt{1 + \sin^2 t}\, dt \;<\; \sqrt{2}\pi.$$

Area. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$. Let the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ be θ. The area of the parallelogram spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ be $A = |\mathbf{w}_1 \times \mathbf{w}_2|$. Then

$$\begin{aligned} \langle \mathbf{w}_1, \mathbf{w}_2 \rangle &= |\mathbf{w}_1||\mathbf{w}_2|\cos\theta, \\ A &= |\mathbf{w}_1||\mathbf{w}_2|\sin\theta, \end{aligned}$$

so

$$A^2 \;=\; |\mathbf{w}_1|^2|\mathbf{w}_2|^2 - \langle \mathbf{w}_1, \mathbf{w}_2 \rangle^2 \;=\; \det\begin{pmatrix} \langle \mathbf{w}_1, \mathbf{w}_1 \rangle & \langle \mathbf{w}_1, \mathbf{w}_2 \rangle \\ \langle \mathbf{w}_2, \mathbf{w}_1 \rangle & \langle \mathbf{w}_2, \mathbf{w}_2 \rangle \end{pmatrix}$$

We see that the area of the parallelogram spanned by $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ is

$$\sqrt{\det \begin{pmatrix} \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle \\ \left\langle \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle \end{pmatrix}} = \sqrt{\det \begin{pmatrix} E & F \\ F & G \end{pmatrix}}$$

Definitions from Do Carmo. If S is a regular surface, a *regular domain* of S is an open, connected subset of S whose boundary is a piecewise smooth homeomorphism of the circle. A *region* of S is the union of a regular domain and its boundary, which should be contained in S.

Now suppose that $\mathbf{r} : U \to S$ is a parameterization, and $Q \subset U$ is a region of U. Then $R = \mathbf{r}(Q)$ is a region of S. We define the area of R to be

$$\int_Q \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv = \int_Q \sqrt{\det \begin{pmatrix} E & F \\ F & G \end{pmatrix}} dudv.$$

Lemma. (Do Carmo.) The area of a region does not depend on the parameterization.

In general we can define the area of a subset of S which is a finite union of regions.

Example. (Do Carmo) Calculate the area of the torus $(\sqrt{x^2 + y^2} - a)^2 + z^2 = R^2$ **where** $R > a$**.**