LECTURE 13: THE GAUSS MAP.

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Diagonalization of self-adjoint linear maps. Let V be a plane in $\mathbb{R}^3$ containing the origin. A *linear map* $T : V \to V$ is a map which satisfies

$$T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v}, \qquad\qquad T(\lambda\mathbf{u}) = \lambda T\mathbf{u}.$$

If $\mathbf{w}_1$, $\mathbf{w}_2$ spans V then the matrix for T in the basis $\mathbf{w}_1$, $\mathbf{w}_2$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if

$$\begin{array}{cc} T\mathbf{w}_1 & T\mathbf{w}_2 \\ \| & \| \\ a\mathbf{w}_1 & b\mathbf{w}_1 \\ + & + \\ c\mathbf{w}_2 & d\mathbf{w}_2 \end{array} \qquad \text{i.e.} \qquad (T\mathbf{w}_1, T\mathbf{w}_2) = (\mathbf{w}_1, \mathbf{w}_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$T(\mu_1\mathbf{w}_1 + \mu_2\mathbf{w}_2) = \mu_1(a\mathbf{w}_1 + c\mathbf{w}_2) + \mu_2(b\mathbf{w}_1 + d\mathbf{w}_2) = (a\mu_1 + b\mu_2)\mathbf{w}_1 + (c\mu_1 + d\mu_2)\mathbf{w}_2$$

So writing vectors in coordinates relative to $\mathbf{w}_1$, $\mathbf{w}_2$,

$$T : \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

If the basis $\mathbf{w}_1$, $\mathbf{w}_2$ is orthonormal then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \langle T\mathbf{w}_1, \mathbf{w}_1\rangle & \langle T\mathbf{w}_2, \mathbf{w}_1\rangle \\ \langle T\mathbf{w}_1, \mathbf{w}_2\rangle & \langle T\mathbf{w}_2, \mathbf{w}_2\rangle \end{pmatrix}.$$

We say T is *self-adjoint* if

$$\langle T\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, T\mathbf{v}\rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$. This is true if and only if the matrix of T relative to every orthonormal basis satisfies $b = c$.

$\mathbf{e}$ is an *eigenvector* of T with *eigenvalue* λ if $\mathbf{e} \neq 0$ and

$$T\mathbf{e} = \lambda\mathbf{e}.$$

Theorem. Suppose $T : V \to V$ is a self adjoint linear map of the plane V. Then there exist orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$ which are eigenvectors for T.

Before giving the proof we note that the matrix of T in the basis $\mathbf{e}_1, \mathbf{e}_2$ is the diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues

$$T\mathbf{e}_1 = \lambda_1\mathbf{e}_1, \qquad\qquad T\mathbf{e}_2 = \lambda_2\mathbf{e}_2.$$

Furthermore assume $\lambda_2 \leq \lambda_1$. Writing the general unit vector in V as

$$\mathbf{w} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,$$

we have

$$\langle T\mathbf{w}, \mathbf{w}\rangle = \langle \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \,,\, \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2\rangle = \cos^2\theta\lambda_1 + \sin^2\theta\lambda_2,$$

so

$$(*) \qquad \langle T\mathbf{e}_2, \mathbf{e}_2\rangle = \lambda_2 \leq \langle T\mathbf{w}, \mathbf{w}\rangle \leq \lambda_1 = \langle T\mathbf{e}_1, \mathbf{e}_2\rangle.$$

Proof of the Theorem. There are two very different proofs of this result. One uses (*) but involves a little analysis. For the other one we first note that if M is a 2×2 matrix then there exists a non-zero vector v with $Mv = 0$, if and only if $\det M = 0$. Indeed, suppose

$$ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -d \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If neither of these gives a non-zero vector v, i.e. if

$$\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and $Mv = 0$ for every v. Conversely if $M\mathbf{v} = 0$ with $\mathbf{v} \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where WLOG $v \neq 0$. Then

$$\begin{matrix} au + bv = 0 \\ cu + dv = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} cau + cbv = 0 \\ acu + adv = 0 \end{matrix} \quad \Rightarrow \quad (ad - bc)v = 0 \quad \Rightarrow \quad ad - bc = 0.$$

Now suppose that $T : V \to V$ is self-adjoint and choose an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for V. In this basis V has matrix

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Now there exists a non-zero eigenvector of T with eigenvalue λ if and only if there exist coordinates $(u, v) \neq (0, 0)$ with

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} a - \lambda & b \\ b & c - \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is true if and only if $(a - \lambda)(c - \lambda) - b^2 = 0$. This quadratic equation either has two real roots $\lambda_1 < \lambda_2$, or $b = 0$ and $a = c$ in which case it has one real root λ, and M is λ times the identity. In the case of two distinct roots, we get one eigenvector for each root,

$$T\mathbf{e}_1 = \lambda_1 \mathbf{e}_1, \qquad T\mathbf{e}_2 = \lambda_2 \mathbf{e}_2.$$

The eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal. Indeed,

$$\lambda_1 \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \ = \ \langle T\mathbf{e}_1, \mathbf{e}_2 \rangle \ = \ \langle \mathbf{e}_1, T\mathbf{e}_2 \rangle \ = \ \lambda_2 \langle \mathbf{e}_1, \mathbf{e}_2 \rangle.$$

Hence

$$(\lambda_1 - \lambda_2) \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \ = \ 0.$$

Since $\lambda_1 \neq \lambda_2$, this implies $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 0$.

The Gauss Map. Suppose that S is a regular surface. At each point $\mathbf{p}$ of S there are two unit normals, $\mathbf{N}$ and $-\mathbf{N}$. If we can choose $\mathbf{N}$ to vary continuously on the whole of S then S is called *orientable.* Examples are the sphere, the torus, the cylinder. Otherwise, S is non-orientable. An example is the Möbius band. In fact if a unit normal varies continuously then it varies smoothly, and finding a smoothly varying unit normal *locally* can always be done. Indeed, if $\mathbf{r} : U \to S$ is a parameterization with $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ then define

$$(*) \qquad \mathbf{N}_0(\mathbf{r}(u,v)) = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|} = \frac{(y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v)}{|(y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v)|}.$$

Then $\mathbf{N}_0$ is a unit normal. Write S^2 for the sphere $x^2 + y^2 + z^2 = 1$. It is clear that the map $\mathbf{N}_0 : \mathbf{r}(U) \to S^2$ is smooth on $\mathbf{r}(U)$.

If $\mathbf{N}$ is a continuous unit normal defined on S and U is connected, then the function $\sigma(u,v) = \langle \mathbf{N}_0(\mathbf{r}(u,v)), \mathbf{N}(\mathbf{r}(u,v))\rangle$ defines a continuous function on U which takes the values ± 1. However, it must be constant for otherwise $\sigma^{-1}(1)$ and $\sigma^{-1}(-1)$ are open non-empty disjoint sets whose union is U which contradicts U being connected. Thus $\mathbf{N} = \mathbf{N}_0$ or $\mathbf{N} = -\mathbf{N}_0$ on the whole set $\mathbf{r}(U)$. This shows that $\mathbf{N}$ is actually smooth.

Now suppose S is a regular orientable surface, so there is a smooth choice of unit normal $\mathbf{N}$. The smooth map

$$\mathbf{N} : S \to S^2$$

is called the Gauss map. For $\mathbf{p} \in S$, the derivative $d\mathbf{N}_{\mathbf{p}}$ is a linear map from $T_{\mathbf{p}}S$ to $T_{\mathbf{N}(\mathbf{p})}S^2$. However, the normal to the unit normal to the sphere $x^2 + y^2 + z^2 = 1$ is (x, y, z), so the normal to S^2 at $\mathbf{N}(\mathbf{p})$ is $\mathbf{N}(\mathbf{p})$, and $T_{\mathbf{p}}S$ and $T_{\mathbf{N}(\mathbf{p})}S^2$ are parallel, and can be identified. So we can think of $d\mathbf{N}$ as a linear map from $T_{\mathbf{p}}S$ to itself.

Lemma. (Do Carmo.) The linear map

$$d\mathbf{N}_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$$

is self-adjoint.

We find that $-d\mathbf{N}_{\mathbf{p}}$ has two eigenvalues $k_1 \geq k_2$ called the *principal curvatures* at $\mathbf{p}$. The product $K = k_1 k_2 = \det d\mathbf{N}_{\mathbf{p}}$ is the Gauss curvature and the average $H = (k_1 + k_2)/2 = -\operatorname{trace} d\mathbf{N}_{\mathbf{p}}$ is the *mean curvature.*