

## LECTURE 13: THE GEOMETRY OF THE GAUSS MAP.

November 7, 2001

**CORRECTION:** To adopt Do Carmo's sign convention, the second fundamental form is defined as  $II(\mathbf{u}, \mathbf{v}) = -d\mathbf{N}(\mathbf{u}, \mathbf{v})$  and the principal curvatures  $k_1 \geq k_2$  are the eigenvalues of  $-d\mathbf{N}$  rather than  $+d\mathbf{N}$ . With these conventions, if the curve bends TOWARDS  $\mathbf{N}$  then  $II(\mathbf{w}, \mathbf{w}) \geq 0$ , while if the curve bends AWAY from  $\mathbf{N}$ ,  $II(\mathbf{w}, \mathbf{w}) \leq 0$ .

**Last Time:** If  $S$  is an orientable regular surface then there exists a smoothly varying normal  $\mathbf{N}$ . The map

$$\mathbf{N} : S \rightarrow S^2$$

is called the *Gauss map*. (In fact, there are two choices of smoothly varying normal,  $\mathbf{N}$  and  $-\mathbf{N}$ , called the *orientations* of  $S$ .)

Then for  $\mathbf{p} \in S$ ,

$$d\mathbf{N}_{\mathbf{p}} : T_{\mathbf{p}}S \rightarrow T_{\mathbf{N}(\mathbf{p})}S^2 \cong T_{\mathbf{p}}S$$

is a self-adjoint linear map. The eigenvalues  $k_1 \leq k_2$  of  $-d\mathbf{N}_{\mathbf{p}}$  are called the *principal curvatures*. The corresponding orthonormal eigenvectors  $\mathbf{e}_1, \mathbf{e}_2$  are the *principal directions*.

$$K = k_1 k_2 = \det d\mathbf{N}_{\mathbf{p}} = \text{Gauss Curvature.}$$

$$H = k_1 + k_2 = \text{Mean Curvature.}$$

**Definition.** The symmetric bilinear form on  $T_{\mathbf{p}}S$  defined by

$$II(u, v) = -\langle d\mathbf{N}_{\mathbf{p}}u, v \rangle$$

is called the *second fundamental form* on  $S$ . We have that if  $\mathbf{w} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$  is a unit vector, then (Euler's formula)

$$II(\mathbf{w}, \mathbf{w}) = -\langle d\mathbf{N}_{\mathbf{p}}\mathbf{w}, \mathbf{w} \rangle = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

$$II(\mathbf{e}_1, \mathbf{e}_1) = k_1 \geq II(\mathbf{w}, \mathbf{w}) \geq II(\mathbf{e}_2, \mathbf{e}_2).$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

**Question.** Geometrically,  $I(\mathbf{w}, \mathbf{w}) = \langle \mathbf{w}, \mathbf{w} \rangle$  is the length squared of  $\mathbf{w}$ . What does  $II(\mathbf{w}, \mathbf{w}) = \langle d\mathbf{N}_{\mathbf{p}}\mathbf{w}, \mathbf{w} \rangle$  mean geometrically?

**Answer.** Certainly it scales like length<sup>2</sup>, so assume that  $|\mathbf{w}| = 1$ . Take the plane through  $\mathbf{p}$  containing vectors  $\mathbf{N}$  and  $\mathbf{w}$ . Around  $\mathbf{p}$ , this intersects  $S$  in the trace of a regular curve whose unit tangent vector at  $\mathbf{p}$  is  $\mathbf{w}$ . If  $\kappa$  is the curvature and  $\mathbf{n}$  is the (Frenet) normal to this curve at  $\mathbf{p}$  then  $\mathbf{n} = \pm\mathbf{N}$ , and

$$II(\mathbf{w}, \mathbf{w}) = -\langle d\mathbf{N}_{\mathbf{p}}\mathbf{w}, \mathbf{w} \rangle = \kappa\langle \mathbf{n}, \mathbf{N} \rangle.$$

**NOTE:** WITH THIS CONVENTION, IF THE CURVE BENDS TOWARDS  $\mathbf{N}$  THEN  $II(\mathbf{w}, \mathbf{w}) \geq 0$ , WHILE IF THE CURVE BENDS AWAY FROM  $\mathbf{N}$ ,  $II(\mathbf{w}, \mathbf{w}) \leq 0$ .

**Definition.** If  $\alpha : (a, b) \rightarrow S$  is a regular curve and  $\mathbf{p} = \alpha(s_0)$  parameterized by arclength, then the *normal curvature* of  $\alpha$  at  $\mathbf{p}$  is

$$\kappa(s_0)\langle \mathbf{n}(s_0), \mathbf{N}(\mathbf{p}) \rangle = \langle \alpha''(s_0), \mathbf{N}(\mathbf{p}) \rangle.$$

**Meusnier's Theorem. (Do Carmo)** If  $S$  is a regular surface,  $\alpha : (a, b) \rightarrow S$  is a smooth curve parameterized by arclength with  $\alpha(s_0) = \mathbf{p}$ , and  $\alpha'(s_0) = \mathbf{t}$ , then

$$\begin{aligned} II(\mathbf{t}, \mathbf{t}) &= -\langle d\mathbf{N}_{\alpha(s_0)}\alpha'(s_0), \alpha'(s_0) \rangle = -\langle (\mathbf{N}\alpha)'(s_0), \alpha'(s_0) \rangle \\ &= -\frac{d}{ds}\bigg|_{s_0} \langle (\mathbf{N}\alpha)(s_0), \alpha'(s_0) \rangle + \langle \mathbf{N}\alpha(s_0), \alpha''(s_0) \rangle = \text{normal curvature of } \alpha \text{ at } \mathbf{p}. \end{aligned}$$

By observation, we calculate the Gauss map, principal curvatures, mean curvature and Gauss curvature for the plane ( $k_1 = k_2 = 0$ ), sphere of radius  $R$ , (with the *inward* pointing normal,  $k_1 = k_2 = 1/R$ ), and cylinder of radius  $R$ , (with the *inward* pointing normal,  $k_1 = 1/R$ ,  $k_2 = 0$ ).

We calculate the Gauss map and principal curvatures for the hyperbolic paraboloid (saddle)  $z = x^2 - y^2$  at the origin (see Do Carmo).

**Definition.** A point  $\mathbf{p}$  in the regular surface  $S$  is *umbilic* if  $k_1 = k_2$  at  $\mathbf{p}$ .

**Theorem. (See Do Carmo).** If all points of a connected surface  $S$  are umbilical points, then  $S$  is either contained in a sphere or in a plane.