LECTURE 15: CALCULATING THE GAUSS MAP IN COORDINATES.

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Last Time: If S is an orientable regular surface then there exists a smoothly varying normal $\mathbf{N}$. The map

$$\mathbf{N} : S \to S^2$$

is called the *Gauss map.*
For $\mathbf{p} \in S$,

$$d\mathbf{N_p} : T_\mathbf{p}S \to T_{\mathbf{N(p)}}S^2 \;\cong\; T_\mathbf{p}S$$

is a self-adjoint linear map. The eigenvalues $k_1 \geq k_2$ of $-d\mathbf{N_p}$ are called the *principal curvatures.* The corresponding orthonormal eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ are the *principal directions.* If $\alpha : (a, b) \to S$ is a regular curve and $\mathbf{p} = \alpha(s_0)$ parameterized by arclength, then the *normal curvature* of α at $\mathbf{p}$ is

$$\kappa(s_0)\langle \mathbf{n}(s_0), \mathbf{N}(\mathbf{p})\rangle \;=\; \langle \alpha''(s_0), \mathbf{N}(\mathbf{p})\rangle \;=\; \langle \alpha'(s_0), dN_\mathbf{p}(\alpha'(s_0))\rangle \;=\; II(\alpha'(s_0), \alpha'(s_0)).$$

CORRECTION: To adopt Do Carmo's sign convention, the second fundamental form is defined as $II(\mathbf{u}, \mathbf{v}) = -\langle d\mathbf{N}(\mathbf{u}), \mathbf{v}\rangle$ and the principal curvatures $k_1 \geq k_2$ are the eigenvalues of $-d\mathbf{N}$ rather than $+d\mathbf{N}$. With these conventions, if the curve bends TOWARDS $\mathbf{N}$ then $II(\mathbf{w}, \mathbf{w}) \geq 0$, while if the curve bends AWAY from $\mathbf{N}$, $II(\mathbf{w}, \mathbf{w}) \leq 0$.

Problem A. Given a parameterization $\mathbf{r} : U \to S$ of a regular surface, compute the matrix of the second fundamental form with respect to the basis associated to $\mathbf{r}$:

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} \;=\; \begin{pmatrix} II(\mathbf{r}_u, \mathbf{r}_u) & II(\mathbf{r}_u, \mathbf{r}_v) \\ II(\mathbf{r}_v, \mathbf{r}_u) & II(\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix}.$$

Problem B. Use this to compute the matrix of dN,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

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defined by

$$\begin{aligned} d\mathbf{N}(\mathbf{r}_u) &= a_{11}\mathbf{r}_u + a_{21}\mathbf{r}_v \\ d\mathbf{N}(\mathbf{r}_v) &= a_{12}\mathbf{r}_u + a_{22}\mathbf{r}_v. \end{aligned}$$

Notation.

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

$$\mathbf{N}_u = d\mathbf{N}\left(\frac{\partial \mathbf{r}}{\partial u}\right) = \frac{\partial \mathbf{N}\circ\mathbf{r}}{\partial u}, \qquad \mathbf{N}_v = d\mathbf{N}\left(\frac{\partial \mathbf{r}}{\partial v}\right) = \frac{\partial \mathbf{N}\circ\mathbf{r}}{\partial v}.$$

Problem A is helped by the equations

$$\begin{aligned} II(\mathbf{r}_u,\mathbf{r}_u) = -\langle d\mathbf{N}\mathbf{r}_u,\mathbf{r}_u\rangle = -\langle \mathbf{N}_u,\mathbf{r}_u\rangle = \langle \mathbf{N},\mathbf{r}_{uu}\rangle. \\ II(\mathbf{r}_u,\mathbf{r}_v) = -\langle d\mathbf{N}\mathbf{r}_u,\mathbf{r}_v\rangle = -\langle \mathbf{N}_u,\mathbf{r}_v\rangle = \langle \mathbf{N},\mathbf{r}_{uv}\rangle. \\ II(\mathbf{r}_v,\mathbf{r}_v) = -\langle d\mathbf{N}\mathbf{r}_v,\mathbf{r}_v\rangle = -\langle \mathbf{N}_v,\mathbf{r}_v\rangle = \langle \mathbf{N},\mathbf{r}_{vv}\rangle. \end{aligned}$$

Problem 2 is solved by taking the scalar product with $\mathbf{r}_u$ and $\mathbf{r}_v$ to get

$$\begin{aligned} -e &= \langle d\mathbf{N}(\mathbf{r}_u),\mathbf{r}_u\rangle &= a_{11}E + a_{21}F \\ -f &= \langle d\mathbf{N}(\mathbf{r}_u),\mathbf{r}_v\rangle &= a_{11}F + a_{21}G \\ -f &= \langle d\mathbf{N}(\mathbf{r}_v),\mathbf{r}_u\rangle &= a_{12}E + a_{22}F \\ -g &= \langle d\mathbf{N}(\mathbf{r}_v),\mathbf{r}_v\rangle &= a_{12}F + a_{22}G. \end{aligned}$$

This can be written as

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

From this we get

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Before proceeding with examples, we gives some definitions which help us to characterize surfaces. A point on a surface is either *elliptic, hyperbolic, parabolic* or *planar.*
Elliptic: $K > 0$ (e.g. the sphere, more generally the ellipsoid.)
Hyperbolic: $K < 0$ (e.g. the saddle $z = xy$.)
Parabolic: $K = 0$, but one of the principal curvatures is non-zero. (e.g. the cylinder, the origin for $z = x^2 \pm z^4$.)

Planar: $K = 0$, and both of the principal curvatures are zero. (e.g. points of the plane, the origin for $z = x^4 \pm y^4$.)

In addition, a point is **umbilical** if $k_1 = k_2$. Of course only elliptic or planar points can be umbilical.
A tangent vector $\mathbf{w}$ is **asymptotic** if $II(\mathbf{w}, \mathbf{w}) = 0$. Note that there are no asymptotic directions at an elliptic point.
A regular curve $\alpha : (a, b) \to S$ is a **line of curvature** if its tangent vector is always in a principal direction, that is

$$(N \circ \alpha)'(t) = \lambda(t)\alpha'(t),$$

where $\lambda : (a, b) \to \mathbb{R}$ is a smooth function.
A regular curve is an **asymptotic curve** if its tangent vector is always an asymptotic direction.

Example 1. Calculate the principal curvatures for the torus given by the parameterization

$$\mathbf{r} : (u, v) \;\to\; ((a + r\cos u)\cos v, (a + r\cos u)\sin v), r\sin u), \qquad \text{where } 0 < a < r.$$

Following Do Carmo,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & (a + r\cos u)^2 \end{pmatrix},$$

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & \cos u(a + r\cos u) \end{pmatrix}.$$

So

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{(a+r\cos u)^2} \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \cos u(a + r\cos u) \end{pmatrix} = -\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos u}{a+r\cos u} \end{pmatrix}.$$

Hence the principal curvatures are

$$\frac{1}{r}, \qquad \frac{\cos u}{a + r\cos u},$$

the Gauss curvature is

$$\frac{\cos u}{r(a + r\cos u)}.$$

So points with $0 < u < \pi$ are elliptic, points with $\pi < u < 2\pi$ are hyperbolic, points with $u = 0, \pi$ are parabolic. There are no umbilic points. The lines u constant or v constant are lines of curvature.