LECTURE 16: CALCULATING THE GAUSS MAP IN COORDINATES. ISOMETRIES.

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Last Time: If S is an orientable regular surface then there exists a smoothly varying normal $\mathbf{N}$. The map

$$\mathbf{N} : S \to S^2$$

is the *Gauss map.*

$$II(\mathbf{u}, \mathbf{v}) = \langle -d\mathbf{N}(\mathbf{u}), \mathbf{v} \rangle.$$

$$\mathbf{r} : U \to S$$

is a parameterization. The matrix of the second fundamental form in the basis $\mathbf{r}_u, \mathbf{r}_v$:

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \mathbf{N}, \mathbf{r}_{uu} \rangle & \langle \mathbf{N}, \mathbf{r}_{uv} \rangle \\ \langle \mathbf{N}, \mathbf{r}_{vu} \rangle & \langle \mathbf{N}, \mathbf{r}_{vv} \rangle \end{pmatrix}.$$

The matrix of dN is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = - \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Example 2. Calculate the principal curvatures for the surface of revolution given by revolving the curve

$$(\phi(u), 0, \psi(u)), \qquad (\phi')^2 + (\psi')^2 = 1,$$

parameterized by arclength, around the z axis.

We have the parameterization

$$\mathbf{r} : (u, v) \ \to \ (\phi(u) \cos v, \phi(u) \sin v, \psi(u)), \qquad \text{where } \phi > 0 \text{ and } 0 < v < 2\pi.$$

Following Do Carmo (we reverse u and v to agree with the previous example),

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2 \end{pmatrix},$$

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \phi'\psi'' - \psi'\phi'' & 0 \\ 0 & \phi\psi' \end{pmatrix}.$$

So

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & \phi^{-2} \end{pmatrix}\begin{pmatrix} \phi'\psi'' - \psi'\phi'' & 0 \\ 0 & \phi\psi' \end{pmatrix} = -\begin{pmatrix} \phi'\psi'' - \psi'\phi'' & 0 \\ 0 & \phi^{-1}\psi' \end{pmatrix}$$

Hence the principal curvatures are

$$\phi'\psi'' - \psi'\phi'' \qquad \frac{\psi'}{\phi},$$

the Gauss curvature is

$$\frac{\psi'(\phi'\psi'' - \psi'\phi'')}{\phi} = -\frac{\phi''}{\phi},$$

Points are elliptic at values of v for which $(\phi, 0, \psi)$ is locally a strictly concave graph over the z axis, and points are hyperbolic at values of v for which this curve is locally a strictly convex graph. One principal curvature measures the signed curvature of the curve, and the other measures whether z increases or decreases. Planar points occur when the tangent to the curve is perpendicular to the z-axis *and* the curvature of the curve vanishes. The curves u =constant and v =constant are the lines of curvature.

Example 3. Following Do Carmo, we compute the principal curvatures for the graph

$$\mathbf{r}(x, y) = (x, y, h(x, y))$$

and we find that the Gaussian curvature is

$$\frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}.$$

Geometric Interpretations. 1. In a neighborhood of a point $\mathbf{p}$, a regular surface S is locally a graph over the tangent plane $T_{\mathbf{p}}S$, and can be rotated and translated to the form

$$z = \frac{k_1x^2 + k_2y^2}{2} + \varepsilon(x, y),$$

where ε is a smooth function with

$$|\varepsilon(x, y)| \leq C|(x, y)|^3.$$

If $\mathbf{p}$ is elliptic or hyperbolic then there exists a parameterization $\mathbf{r} : U \to S$ with $\mathbf{r}(0,0) = \mathbf{p}$ so that (after the rotation and translation) locally the intersection of S with the plane with z constant is

$$\frac{k_1 u^2 + k_2 v^2}{2} = z.$$

In particular if $\mathbf{p}$ is elliptic then locally S lies to one side of the tangent plane, while if $\mathbf{p}$ is hyperbolic then locally S lies on both sides of the tangent plane.

2. At $\mathbf{p}$, the Gauss curvature satisfies

$$|K| \;=\; \lim_{\substack{\text{diameter}(A)\to 0\\ A\ni\mathbf{p}}} \frac{\text{area}\,\mathbf{N}(A)}{\text{area}\,A}.$$

The Gauss curvature is positive if $\mathbf{N}$ preserves orientation close to $\mathbf{p}$, and negative if it reverses orientation.

3. At $\mathbf{p}$, the mean curvature satisfies

$$|H| \;=\; \lim_{\substack{\text{diameter}(A)\to 0\\ A\ni\mathbf{p}}} \quad \lim_{\varepsilon\to 0} \frac{\text{volume}\{\mathbf{q} + t\mathbf{N}(\mathbf{q}) : \mathbf{q} \in A, 0 \le t \le \varepsilon\}}{\varepsilon\,\text{area}\,A}.$$

To see 1, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the principal directions at $\mathbf{p}$ and rotate and translate so that $\mathbf{p} \to (0,0,0)$, $\mathbf{e}_1 \to (1,0,0)$ and $\mathbf{e}_2 \to (0,1,0)$. After this, the tangent plane is now the xy plane. Given a parameterization $\bar{\mathbf{r}}$ of S with $\bar{\mathbf{r}}(0,0) = (0,0,0)$, if

$$\bar{\mathbf{r}}(\bar{u},\bar{v}) = (x(\bar{u},\bar{v}), y(\bar{u},\bar{v}), z(\bar{u},\bar{v}))$$

then writing

$$\pi(x,y,z) = (x,y),$$

the composition $\pi \circ \bar{\mathbf{r}}$ is smooth and $d(\pi \circ \bar{\mathbf{r}})$ at $(0,0)$ is one-to-one an so by the inverse function theorem, locally

$$\pi \circ \bar{\mathbf{r}}(\bar{u},\bar{v}) = (x(\bar{u},\bar{v}), y(\bar{u},\bar{v}))$$

is invertible, so

$$(x,y) \to (\bar{u}(x,y), \bar{v}(x,y))$$

is smooth and S is locally parameterized by

$$\mathbf{r}(x,y) = (x, y, z(\bar{u}(x,y), \bar{v}(x,y)) = (x, y, h(x,y))$$

where

$$h(x,y) = z(\bar{u}(x,y), \bar{v}(x,y)).$$

Now the multi-variable version Taylor's theorem gives

$$h(x,y) = h(0,0) + h_x(0,0)x + h_y(0,0)y + \frac{h_{xx}(0,0)x^2 + 2h_{xy}(0,0)xy + h_{yy}(0,0)y^2}{2} \; + \; \varepsilon(x,y)$$

where the error $\varepsilon(x,y)$ is bounded by $C|(x,y)|^3$ for some constant C. This is proved from the one variable version applied to $H(s) = h(sx, sy)$ at $s = 1$, and the chain rule.
Now $\mathbf{r}(0,0) = (0,0,0)$, so $h(0,0) = 0$. Furthermore, the tangent plane to S at $(0,0,0)$ is the xy plane and $\mathbf{r}_x = (1, 0, h_x)$, so $h_x(0,0) = 0$ and similarly $h_y(0,0) = 0$. We also have $\mathbf{N}(0,0) = (0,0,1)$ and so

$$\begin{aligned} h_{xx}(0,0) &= \langle \mathbf{N}, \mathbf{r}_{xx} \rangle = \langle -d\mathbf{N}\mathbf{r}_x, \mathbf{r}_x \rangle = k_1, \\ h_{xy}(0,0) &= \langle \mathbf{N}, \mathbf{r}_{xy} \rangle = \langle -d\mathbf{N}\mathbf{r}_x, \mathbf{r}_y \rangle = 0, \\ h_{yy}(0,0) &= \langle \mathbf{N}, \mathbf{r}_{yy} \rangle = \langle -d\mathbf{N}\mathbf{r}_y, \mathbf{r}_y \rangle = k_2. \end{aligned}$$

Hence

$$h(x,y) = \frac{k_1 x^2 + k_2 y^2}{2} \; + \; \varepsilon(x,y),$$

where k_1, k_2 are evaluated at $(0,0)$.

The *Morse Lemma* (proved using Taylor's theorem and the inverse function theorem) shows that if $k_1 \neq 0$ and $k_2 \neq 0$, there exist smooth functions $a(x,y)$ and $b(x,y)$ with $a(0,0) = b(0,0) = 0$, such that

$$u = x(1 + +a(x,y)), \qquad v = y(1 + b(x,y)),$$

satisfy

$$h(x,y) = \frac{k_1 u^2 + k_2 v^2}{2},$$

and close to $(0,0)$ the map $(x,y) \to (u,v)$ is invertible with smooth inverse.