LECTURES 18: LOCAL ISOMETRIES.

November 28, 2001

Last Time:
Theorem. K can be written in terms of E, F, G.
A note on the modern approach: The curve $\alpha(\theta) = \frac{1}{2}(\cos\theta, \sin\theta, \sqrt{3})$ is in the surface S^2. The vector field $\mathbf{v}(x, y, z) = (-xz, xz, 0)$ is tangent to S^2.
Question: What is

$$\mathbf{v}_\theta = \frac{d\mathbf{v}}{d\theta}? \qquad \left(\text{more precisely } \frac{d\mathbf{v}\circ\alpha}{d\theta}?\right)$$

Answer:

$$\frac{d\frac{\sqrt{3}}{4}(-\sin\theta, \cos\theta, 0)}{d\theta} = \frac{\sqrt{3}}{4}(-\cos\theta, -\sin\theta, 0).$$

Question: Is this tangent to S^2 along the curve α?
Answer: No!
Question: Can we define a way to differentiate tangent fields along curves and get something which is again tangent to the surface?
Answer: Yes! First differentiate then project to the tangent plane:

$$\begin{aligned} D_\theta\mathbf{v} = \frac{D\mathbf{v}}{d\theta} &= \frac{\sqrt{3}}{4}(-\cos\theta, -\sin\theta, 0) - \langle\frac{\sqrt{3}}{4}(-\cos\theta, -\sin\theta, 0), \mathbf{N}\rangle\mathbf{N} \\ &= \frac{\sqrt{3}}{4}(-\cos\theta, -\sin\theta, 0) - \frac{\sqrt{3}}{16}\langle(-\cos\theta, -\sin\theta, 0), (\cos\theta, \sin\theta, \sqrt{3})\rangle(\cos\theta, \sin\theta, \sqrt{3}) \\ &= \frac{\sqrt{3}}{16}(-3\cos\theta, -3\sin\theta, \sqrt{3}) \end{aligned}$$

This is the *covariant derivative* and it can be computed in a parameterization using E, F, G. Indeed, we saw last time that for $\mathbf{r} : U \to S$ a parameterization,

$$D_u\mathbf{r}_u \;=\; \mathbf{r}_{uu} = \langle\mathbf{r}_{uu}, \mathbf{N}\rangle\mathbf{N} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v,$$

and the coefficients Γ^k_{ij} can be computed. In many ways, covariant differentiation behaves like usual differentiation. However, we do not have $D_uD_v = D_vD_u$. Instead,

$$\langle(D_vD_u - D_uD_v)\mathbf{r}_u, \mathbf{r}_v\rangle = (EG - F^2)K.$$

Definition. A diffeomorphism $\phi : S \to \bar{S}$ is an *isometry* if for all $\mathbf{q} \in S$, we have

$$(*) \qquad \langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbf{q}} = \langle d\phi_{\mathbf{q}}(\mathbf{w}_1), d\phi_{\mathbf{q}}(\mathbf{w}_2) \rangle_{\phi(\mathbf{q})}, \qquad \text{for all } \mathbf{w}_1, \mathbf{w}_2 \in T_{\mathbf{q}}S.$$

If there exists an isometry $\phi : S \to \bar{S}$ then S and $\bar{S}$ are called *isometric*.

Note that since the chain rule holds:

$$\phi : S \to \bar{S}, \qquad \psi : \bar{S} \to \bar{\bar{S}}, \quad \Rightarrow \quad d(\psi \circ \phi) = d\psi \circ d\phi,$$

we see that if ϕ is a diffeomorphism then $d(\phi^{-1}) = (d\phi)^{-1}$ and so if ϕ is an isometry then so is ϕ^{-1}.

Definition. S is *locally isometric* to $\bar{S}$ at $\mathbf{p}$ if there exists an open neighborhood of V of $\mathbf{p}$ in S, and an open subset $\bar{V}$ of $\bar{S}$ and an isometry $\phi : V \to \bar{V}$. (The map ϕ is called a local isometry from S to $\bar{S}$ at $\mathbf{p}$.)
The surface S is *locally isometric* to $\bar{S}$, if for every $\mathbf{p} \in S$, S is locally isometric to $\bar{S}$ at $\mathbf{p}$. S and $\bar{S}$ are said to be *locally isometric*, if S is locally isometric to $\bar{S}$, and $\bar{S}$ is locally isometric to S.

Proposition. Suppose $\mathbf{r} : U \to S$ and $\bar{\mathbf{r}} : U \to \bar{S}$ are parameterizations. Then $\bar{\mathbf{r}} \circ \mathbf{r}^{-1}$ is an isometry if and only if $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$.