LECTURE 2: CURVATURE AND TORSION.

September 24, 2001

Curvature. Recall from last time that we consider a general regular curve $\mathbf{r}(s)$ parameterized by arclength. Then

$$\mathbf{t}(s) \;=\; \mathbf{r}'(s)$$

is a unit tangent vector. The *curvature* of the curve at s is

$$\kappa \;=\; |\mathbf{t}'(s)| \;=\; |\mathbf{r}''(s)|.$$

Lemma. The curvature of the circle of radius ρ is $1/\rho$.
Lemma. $\mathbf{t}'(s)$ is perpendicular to $\mathbf{t}(s)$.
From now on we assume that we have a curve with $\kappa \neq 0$ we write

$$\mathbf{n} \;=\; \mathbf{t}'/|\mathbf{t}'|$$

so $\mathbf{n}$ is perpendicular to $\mathbf{t}$ and

$$\mathbf{t}' \;=\; \kappa\mathbf{n}.$$

Problem: Describe curvature geometrically. For this we use Taylor's theorem. Let's start with the geometric interpretation of $\mathbf{t}$. The *tangent line to the curve at* s_0 is the line passing through $\mathbf{r}(s_0)$ in the direction $\mathbf{t}(s_0)$. It is the best straight line approximation to the curve at s_0. Indeed, by Taylor's theorem, for s close to s_0,

$$|\,\mathbf{r}(s) \;-\; (\mathbf{r}(s_0) + \mathbf{r}'(s_0)(s - s_0))\,| \;\leq\; C|s - s_0|^2.$$

The plane containing $\mathbf{r}(s_0)$ spanned by vectors $\mathbf{t}(s_0)$ and $\mathbf{n}(s_0)$ is called the *osculating plane* at s_0. The *osculating* circle at s_0 is the circle in the osculating plane which has radius $1/\kappa(s_0)$ and center $\mathbf{r}(s_0) + \mathbf{n}(s_0)/\kappa(s_0)$. This circle is tangent to the curve $\mathbf{r}$ at s_0, and (when oriented correctly) has the same values of $\mathbf{t}$, $\mathbf{n}$ and κ at $\mathbf{r}(s_0)$. We can check this by computing. Indeed, set $\kappa_0 = \kappa(s_0)$ and parameterize the circle by arclength:

$$\mathbf{u}(s) \;=\; \mathbf{r}(s_0) + \frac{1}{\kappa_0}\mathbf{n}(s_0) \;-\; \frac{1}{\kappa_0}\mathbf{n}\cos\kappa_0(s - s_0) \;+\; \frac{1}{\kappa_0}\mathbf{t}\sin\kappa_0(s - s_0).$$

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We have done this so that $\mathbf{u}(s_0) = \mathbf{r}(s_0)$. We find immediately by differentiating that $\mathbf{u}'(s_0) = \mathbf{r}'(s_0)$ and $\mathbf{u}''(s_0) = \mathbf{r}''(s_0)$ which is eactly what we need.
Now by Taylor's theorem,

$$\left|\mathbf{r}(s) \;-\; \Big(\mathbf{r}(s_0) + \mathbf{t}(s_0)(s - s_0) + \tfrac{1}{2\kappa_0}\mathbf{n}(s_0)(s - s_0)\Big)\right| \;\leq\; C|s - s_0|^3,$$

and

$$\left|\mathbf{u}(s) \;-\; \Big(\mathbf{r}(s_0) + \mathbf{t}(s_0)(s - s_0) + \tfrac{1}{2\kappa_0}\mathbf{n}(s_0)(s - s_0)\Big)\right| \;\leq\; C_1|s - s_0|^3,$$

so

$$|\,\mathbf{r}(s) - \mathbf{u}(s)\,| \;\leq\; (C + C_1)|s - s_0|^3.$$

This shows that the osculating circle for s_0 is the circle which best approximates the curve as s approaches s_0.

Now write

$$\mathbf{b} \;=\; \mathbf{t} \times \mathbf{n}.$$

Lemma. $\mathbf{b}'$ is parallel to $\mathbf{n}$.
Proof. $\mathbf{b}'$ is perpendicular to $\mathbf{b}$. (Just differentiate the equation $\mathbf{b} \cdot \mathbf{b} = 1$.) However it is also perpendicular to $\mathbf{t}$ since

$$0 \;=\; \mathbf{t} \cdot \mathbf{b}$$

so

$$0 \;=\; (\mathbf{t} \cdot \mathbf{b})' \;=\; \mathbf{t}' \cdot \mathbf{b} \;+\; \mathbf{t} \cdot \mathbf{b}' \;=\; \kappa\mathbf{n} \cdot \mathbf{b} \;+\; \mathbf{t} \cdot \mathbf{b}' \;=\; \mathbf{t} \cdot \mathbf{b}'.$$

Hence $\mathbf{b}'$ must be parallel to $\mathbf{n}$.

Definition. We define the *torsion* τ by

$$\mathbf{b}' \;=\; \tau\mathbf{n}.$$

Then τ measures the rate at which the osculating plane changes direction. If the torsion is identically zero, the curve lies in a plane.

Definition. The triple of vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is called the **Frenet** frame at s.

Example 4. The helix $\mathbf{r}(\theta)$ given by

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= a\theta \end{aligned}$$

has constant curvature and torsion.

Fundamental theorem of the local theory of curves. For $\kappa(s) > 0$ and $\tau(s)$ smooth functions on (a, b), there exists a regular curve parameterized by arclength such that $k(s)$ is the curvature and $\tau(s)$ is the torsion. Any other curve having this curvature and torsion function can be obtained from this one by rotating and translating.

Lemma.

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} - \tau \mathbf{b} \\ \mathbf{b}' &= \tau \mathbf{n}. \end{aligned} \tag{*}$$

Proof. We just need to check the second equation. We can use either the cross or dot product. Using the cross product we can write $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ and so

$$\mathbf{n}' = \mathbf{b}' \times \mathbf{t} + \mathbf{b} \times \mathbf{t}' = \tau \mathbf{n} \times \mathbf{t} + \kappa \mathbf{b} \times \mathbf{n} = -\tau \mathbf{b} - \kappa \mathbf{t}.$$

There are two steps in proving the existence in the Fundamental Theorem.
Step 1. Show that (*) can be solved - that is given κ and τ we can find $\mathbf{t}, \mathbf{n}, \mathbf{b}$ - with any initial frame $\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)$.
Step 2. Show that the solution to (*) gives a solution to the Fundamental Theorem if the initial frame starts out right handed and orthonormal.
We will make use of the following Theorem:
Theorem: Solution of ordinary differential equations. Suppose that

$$F : \mathbb{R}^n \times (a, b) \rightarrow \mathbb{R}^n$$

is a continuous function and there exists C such that the following Lipschitz condition is satisfied:

$$|F(\mathbf{u}, s) - F(\mathbf{v}, s)| \leq C\,|\mathbf{u} - \mathbf{v}|, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Then for each $\mathbf{V}_0 \in \mathbb{R}^n$ there exists a unique function $\mathbf{V} : (a, b) \rightarrow \mathbb{R}^n$ solving

$$\begin{aligned} \frac{d\mathbf{V}}{ds} &= F(\mathbf{V}(s), s), \\ \mathbf{V}(s_0) &= \mathbf{V}_0. \end{aligned} \tag{3}$$

If F is k times differentiable then $\mathbf{V}$ is $k + 1$ times differentiable

Idea of the Proof: Picard iteration. A solution $\mathbf{V}$ of equations (3) satisfies

$$\mathbf{V}(s) = \mathbf{V}_0 + \int_{s_0}^{s} F(\mathbf{V}(\sigma), \sigma)\, d\sigma. \tag{5}$$

We define a sequence of functions $\mathbf{V}^{(m)}$ by

$$\begin{aligned} \mathbf{V}^{(0)}(s) &= \mathbf{V}_0, \qquad \text{for all } s, \\ \mathbf{V}^{(i+1)}(s) &= \mathbf{V}_0 + \int_{s_0}^{s} F(\mathbf{V}^{(i)}(\sigma), \sigma)\, d\sigma. \end{aligned}$$

The problem is to show that $\mathbf{V}^{(i)}(s)$ converges to a limit $\mathbf{V}(s)$ as $i \to \infty$ and the limit function is continuous and satisfies (5). Uniqueness is also proved using (5). Due to the popular vote, we will prove this next time.
Now we put the three 3-vectors

$$\mathbf{t} = (t_1, t_2, t_3), \quad \mathbf{n} = (n_1, n_2, n_3), \quad \mathbf{b} = (b_1, b_2, b_3),$$

together in a 9-vector $\mathbf{V}(s)$. The above Lemma shows that

(1)
$$\frac{d\mathbf{V}(s)}{ds} = \frac{d}{ds} \begin{pmatrix} t_1(s) \\ t_2(s) \\ t_3(s) \\ n_1(s) \\ n_2(s) \\ n_3(s) \\ b_1(s) \\ b_2(s) \\ b_3(s) \end{pmatrix} = F(\mathbf{V}(s), s)$$

where $F : \mathbb{R}^9 \times (a, b) \to \mathbb{R}^9$ is defined by

(2)
$$F \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ s \end{pmatrix} = \begin{pmatrix} \kappa(s) v_4 \\ \kappa(s) v_5 \\ \kappa(s) v_6 \\ -\kappa(s) v_1 - \tau(s) v_4 \\ -\kappa(s) v_2 - \tau(s) v_5 \\ -\kappa(s) v_3 - \tau(s) v_6 \\ \tau(s) v_4 \\ \tau(s) v_5 \\ \tau(s) v_6 \end{pmatrix}.$$

In order to get convergence on the whole interval it is important that $F(\mathbf{v}, s)$ does not grow too fast with $|\mathbf{v}|$.

One dimensional example. $F(V,s) = V^2$. Solve for example

$$\begin{aligned}\frac{dV}{ds} &= V^2,\\ V(0) &= 1.\end{aligned}$$

Then

$$\frac{dV}{V^2} = ds \qquad \Rightarrow \qquad \int_1^V \frac{dV}{V^2} = \int_0^s ds \Rightarrow \qquad 1 - \frac{1}{V} = s \qquad \Rightarrow \qquad V = \frac{1}{1-s}.$$

The solution cannot be extended continuously across $s = 1$.

Proof of Step 2. Pick a right handed orthonormal frame $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ and a position vector $\mathbf{r}_0$. By the existence for ordinary differential equations we can find a solution to (*) with

$$\mathbf{t}(s_0) = \mathbf{t}_0, \qquad \mathbf{n}(s_0) = \mathbf{n}_0, \qquad \mathbf{b}(s_0) = \mathbf{b}_0.$$

Set

$$\mathbf{r}(s) = \mathbf{r}_0 + \int_{s_0}^s \mathbf{t}(\sigma)\, d\sigma.$$

Then $\mathbf{r}(s)$ is a smooth parameterized curve with tangent $\mathbf{t}(s)$. We claim that $|\mathbf{t}| = 1$, that is $\mathbf{r}$ is parameterized by arclength. This will complete Step 2, since then from (*), the curvature and torsion are $\kappa(s)$ and $\tau(s)$. To show that $|\mathbf{t}| = 1$, we use the uniqueness for ordinary differential equations. We have

$$\begin{aligned}(\mathbf{t}\cdot\mathbf{t})'(s) &= 2\kappa\,\mathbf{t}\cdot\mathbf{n}\\ (\mathbf{n}\cdot\mathbf{n})'(s) &= -2\kappa\,\mathbf{t}\cdot\mathbf{n} - 2\tau\,\mathbf{n}\cdot\mathbf{b}\\ (\mathbf{b}\cdot\mathbf{b})'(s) &= 2\tau\,\mathbf{n}\cdot\mathbf{b}\\ (\mathbf{t}\cdot\mathbf{n})'(s) &= \kappa\,\mathbf{n}\cdot\mathbf{n} - \kappa\,\mathbf{t}\cdot\mathbf{n} - \tau\,\mathbf{n}\cdot\mathbf{b}\\ (\mathbf{n}\cdot\mathbf{b})'(s) &= -\kappa\,\mathbf{t}\cdot\mathbf{b} - \tau\,\mathbf{b}\cdot\mathbf{b} + \tau\mathbf{n}\cdot\mathbf{n}\\ (\mathbf{t}\cdot\mathbf{b})'(s) &= \kappa\,\mathbf{n}\cdot\mathbf{b} + \tau\mathbf{t}\cdot\mathbf{n}.\end{aligned} \tag{**}$$

This is an ordinary differential equation. Since we chose $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ to be a right handed orthonormal frame, we have immediately that

$$\begin{pmatrix}\mathbf{t}\cdot\mathbf{t}\\ \mathbf{n}\cdot\mathbf{n}\\ \mathbf{b}\cdot\mathbf{b}\\ \mathbf{t}\cdot\mathbf{n}\\ \mathbf{n}\cdot\mathbf{b}\\ \mathbf{t}\cdot\mathbf{b}\end{pmatrix}(s_0) = \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}.$$

However, one can check that if

$$\begin{pmatrix} \mathbf{t}\cdot\mathbf{t} \\ \mathbf{n}\cdot\mathbf{n} \\ \mathbf{b}\cdot\mathbf{b} \\ \mathbf{t}\cdot\mathbf{n} \\ \mathbf{n}\cdot\mathbf{b} \\ \mathbf{t}\cdot\mathbf{b} \end{pmatrix}(s) \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for all } s$$

then (**) holds, and so by the uniqueness for ordinary differential equations, this is true and $|\mathbf{t}| = 1$.