LECTURE 3: EXISTENCE AND UNIQUENESS FOR ORDINARY DIFFERENTIAL EQUATIONS.

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We will begin by discussing existence and uniqueness for ordinary differential equations and for this we will need some background.

Definition. A *metric space* (X, d) is a set X and a function $d : X \times X \to [0, \infty)$ such that

(1). $d(u, v) = 0 \;\Leftrightarrow\; u = v$.

(2). $d(u, v) = d(v, u)$. (symmetric)

(3). $d(u, w) \;\leq\; d(u, v) + d(v, w)$. (triangle inequality)

Examples.

(1). $\mathbb{R}^n$ with $d(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}|$

(2). Let $\mathbf{u} \in \mathbb{R}^n$ and $R > 0$ and let $B_R(\mathbf{v}_0) = \{\, \mathbf{v} : |\mathbf{v} - \mathbf{v}_0| \leq R \,\}$. For an interval (a, b) and $t_0 \in (a, b)$ define

$$(*) \qquad X \;=\; \{\, \mathbf{v} : (a, b) \to B_R(\mathbf{x}) : \; \mathbf{v} \text{ is continuous and } \mathbf{v}(t_0) = \mathbf{v}_0 \,\}.$$

For $\mathbf{u}, \mathbf{v} \in X$ define

$$d(\mathbf{u}, \mathbf{v}) \;=\; \sup_{t \in (a,b)} |\mathbf{u}(t) - \mathbf{v}(t)|.$$

This distance is called the supremum norm distance. To see the triangle inequality, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$,

$$|\mathbf{u}(t) - \mathbf{w}(t)| \;\leq\; |\mathbf{u}(t) - \mathbf{v}(t)| + |\mathbf{v}(t) - \mathbf{w}(t)| \;\leq\; d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}).$$

Hence taking the supremum,

$$d(\mathbf{u}, \mathbf{w}) \;\leq\; d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}).$$

Definition. A sequence u_i in a metric space X converges to $u \in X$ as $i \to \infty$ if the sequence $d(u_i, u)$ converges to zero as $i \to \infty$, that is for each $\varepsilon > 0$ there exists N such that $d(u_i, u) < \varepsilon$ when $i \geq N$.

Definition. A sequence u_i in a metric space X is *Cauchy* if for each $\varepsilon > 0$ there exists N such that $d(u_i, u_j) < \varepsilon$ when $i, j \geq N$.

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Definition. The metric space X is *complete* if every Cauchy sequence has a limit.

Examples.

(1). The Real numbers $\mathbb{R}$ with the usual distance can be formed from the Rationals $\mathbb{Q}$, by throwing in limits for all the Cauchy sequences of $\mathbb{Q}$. $\mathbb{R}$ is complete. The space $\mathbb{R}^n$ with the usual distance is complete.

(2). The space X in (*) with the supremum norm distance is complete. Indeed, suppose that $\mathbf{v}_i$ is Cauchy in X. Then given $\varepsilon > 0$ there exists N such that $d(\mathbf{v}_i, \mathbf{v}_j) < \varepsilon$ for $i, j \geq N$. But then for each s,

$$|\mathbf{v}_i(t) - \mathbf{v}_j(t)| \;\leq\; d(\mathbf{v}_i, \mathbf{v}_j) \;<\; \varepsilon, \qquad\qquad i, j \geq N.$$

But then $\mathbf{v}_i(t)$ is Cauchy in $\mathbb{R}^n$ and so has a limit $\mathbf{v}(t)$. Moreover,

$$(**) \qquad |\mathbf{v}_i(t) - \mathbf{v}(t)| \;=\; \lim_{j \to \infty} |\mathbf{v}_i(t) - \mathbf{v}_j(t)| \;\leq\; \varepsilon, \qquad\qquad n \geq N.$$

However $\mathbf{v}_N$ is continuous so there exists $\delta > 0$ such that

$$|\mathbf{v}_N(s) - \mathbf{v}_N(t)| < \varepsilon, \qquad\qquad |s - t| < \delta$$

then

$$|\mathbf{v}(s) - \mathbf{v}(t)| \;\leq\; |\mathbf{v}(s) - \mathbf{v}_N(s)| + |\mathbf{v}_N(s) - \mathbf{v}_N(t)| + |\mathbf{v}_N(t) - \mathbf{v}(t)| \;<\; 3\varepsilon, \qquad\qquad |s - t| < \delta.$$

Hence $\mathbf{v}$ is continuous and by (**), $\mathbf{v}_i$ converges to $\mathbf{v}$ as $i \to \infty$. Hence (X, d) is complete.

Lemma. If X is a metric space and u_i converges to u as $i \to \infty$ and $v \in X$ then $d(u_i, v) \to d(u, v)$ as $i \to \infty$.
Proof. From the triangle inequality,

$$|d(u, v) - d(u_i, v)| \leq d(u_i, u) \;\to\; 0, \qquad\qquad i \to \infty.$$

Contraction Mapping Theorem. Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is there exists ρ with $0 \leq \rho < 1$ such that

$$d(Tu, Tv) \;\leq\; \rho d(u, v), \qquad\qquad \text{for all } u, v \in X.$$

Then T has a unique fixed point; there exists $v \in X$ with $Tv = v$ and if $Tu = u$ then $u = v$.

Proof. Pick any $w \in X$ and define a sequence v_i in X by

$$v_0 = w, \qquad\qquad v_i = Tv_{i-1}.$$

Then we claim that v_i is Cauchy. Indeed, if $j > i$ then

$$\begin{aligned}
d(v_1, v_2) &= d(Tv_0, Tv_1) \le \rho\, d(v_0, v_1). \\
d(v_2, v_3) &= d(Tv_1, Tv_2) \le \rho\, d(v_1, v_2) \le \rho^2 d(v_0, v_1). \\
&\vdots \\
d(v_k, v_{k+1}) &= d(Tv_{k-1}, Tv_k) \le \rho\, d(v_{k-1}, v_k) \le \rho^k d(v_0, v_1).
\end{aligned}$$

Hence for $j > i$,

$$d(v_i, v_j) \le \sum_{k=i}^{j-1} d(v_k, v_{k+1}) \le \sum_{k=i}^{j-1} \rho^k d(v_0, v_1) = \frac{\rho^i - \rho^j}{1-\rho} d(v_0, v_1).$$

Since $\rho^i \to 0$ as $i \to \infty$ we find that the sequence v_i is Cauchy and there exists v such that $v_i \to v$ as $i \to \infty$. Since $v_{i+1} = Tv_i$ we have $Tv_i \to v$ as $i \to \infty$ and

$$d(Tv, v) = \lim_{j\to\infty} d(Tv, Tv_j) \le \rho \lim_{j\to\infty} d(v, v_j) = 0.$$

Now suppose that $Tu = u$. Then

$$d(u, v) = d(Tu, Tv) \le \rho\, d(u, v).$$

Hence $(1-\rho)d(u, v) \le 0$ which implies $d(u, v) = 0$ so $u = v$.

Theorem: Solution of ordinary differential equations. Suppose that $R > 0$, $\mathbf{v}_0 \in \mathbb{R}^n$ and s_0 is a point in the interval (a, b). Suppose

$$F : B_R(\mathbf{v}_0) \times (a, b) \to \mathbb{R}^n$$

is a continuous function and there exists C such that the following conditions are satisfied:
Bound:

$$F(\mathbf{u}, s) \le CR,$$

Lipschitz condition:

$$|F(\mathbf{u}, t) - F(\mathbf{v}, t)| \le C\,|\mathbf{u} - \mathbf{v}|, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in B_R(\mathbf{v}_0),\ t \in (a, b).$$

Then for each there exists $\varepsilon > 0$ and a unique continuously differentiable function $\mathbf{V} : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}^n$ solving

$$\text{(***)} \qquad \begin{aligned}
\frac{d\mathbf{v}}{dt} &= F(\mathbf{v}(t), t), \\
\mathbf{v}(t_0) &= \mathbf{v}_0.
\end{aligned}$$

Moreover, if F is k times continuously differentiable then $\mathbf{v}$ is $k+1$ times continuously differentiable.
In fact we can choose any $\varepsilon < 1/C$ with $(t_0 - \varepsilon, t_0 + \varepsilon) \subset (a, b)$.

Correction from last time: If the function F is continuous but not Lipschitz in $\mathbf{v}$ then we get existence of the solution (Peano's Existence Theorem) but uniqueness may not hold. For example, in one dimension

$$\begin{aligned} \frac{dv}{dt} &= 2\sqrt{|v|}, \\ \mathbf{v}(0) &= 0. \end{aligned}$$

has solutions $v = 0$ and $v = t^2$.

Proof. For ε satisfying $(t_0 - \varepsilon, t_0 + \varepsilon) \subset (a, b)$, $\varepsilon < 1/C$, set

$$X \;=\; \{\mathbf{v} : (t_0 - \varepsilon, t_0 + \varepsilon) \;\to\; B_R(\mathbf{V}_0) : \mathbf{v} \text{ continuous }, \mathbf{v}(t_0) = \mathbf{v}_0\},$$

with the supremum norm distance. We showed that X is a complete metric space. Now set

$$(T\mathbf{v})(t) \;=\; \mathbf{V}_0 \;+\; \int_{t_0}^{t} F(\mathbf{v}(\sigma), \sigma)\, d\sigma.$$

Then $\mathbf{v}$ solves (***) if and only if $T\mathbf{v} = \mathbf{v}$. We will show that for $\varepsilon < 1/C$, T is a contraction of X. First we show that T maps X into X. Certainly if $\mathbf{v}$ is continuous then $T\mathbf{v}$ is continuous. Furthermore

$$T : (t_0 - \varepsilon, t_0 + \varepsilon) \to B_R(\mathbf{v}_0).$$

Indeed,

$$|T\mathbf{v}(t) - \mathbf{v}_0| \;=\; \left| \int_{t_0}^{t} F(\mathbf{v}(\sigma), \sigma)\, d\sigma \right| \;\leq\; |t - t_0| CR \;< R.$$

Finally, T is a contraction. Indeed,

$$|T\mathbf{v}(t) - T\mathbf{w}(t)| \;\leq\; \int_{t_0}^{t} |F(\mathbf{v}(\sigma), \sigma) - F(\mathbf{w}(\sigma), \sigma)|\, d\sigma \;\leq\; C|t - t_0| d(\mathbf{v}, \mathbf{w}) \;<\; C\varepsilon d(\mathbf{v}, \mathbf{w}).$$

Thus

$$d(T\mathbf{v}, T\mathbf{w}) \;\leq\; (C\varepsilon) d(\mathbf{v}, \mathbf{w})$$

and T is a contraction. Hence T has a unique fixed point.

If $F : \mathbb{R}^n \times (a, b)$ is continuous and Lipschitz in $\mathbf{v}$ then we can get a solution on the whole interval (a, b) by iterating the result to get a solution on small overlapping intervals of constant length. We can also obtain uniqueness, but we omit the details.