LECTURE 4: RIGID MOTION. REGULAR SURFACES.

October 1, 2001

We recap what happened last week.
Fundamental theorem of the local theory of curves. For $\kappa(s) > 0$ and $\tau(s)$ smooth functions on (a, b), there exists a regular curve parameterized by arclength such that $k(s)$ is the curvature and $\tau(s)$ is the torsion. Any other curve having this curvature and torsion function can be obtained from this one by rotating and translating.

Recall the Lemma:

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} - \tau\mathbf{b} \\ \mathbf{b}' &= \tau\mathbf{n}. \end{aligned} \tag{*}$$

By the existence of solutions to ordinary equations, this can be solved with any values of $\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)$. In particular we can take a right handed orthonormal frame, for example $\mathbf{t}(s_0) = \mathbf{e}_1$, $\mathbf{n}(s_0) = \mathbf{e}_2$, $\mathbf{b}(s_0) = \mathbf{e}_3$. We showed already using the uniqueness of solutions to ordinary differential equations that this remains orthonormal at distance s. In particular $|\mathbf{t}| = 1$ and

$$\mathbf{r}(s) = \int_{s_0}^{s} \mathbf{t}(\sigma)\, d\sigma$$

is a curve parameterized by arclength. Since $|\mathbf{n}| = 1$, the curvature is $\kappa(s)$, and the normal is $\mathbf{n}$. Since $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, the torsion $\tau(s)$.
It remains to show that if

$$\bar{\mathbf{r}} : (a, b) \to \mathbb{R}^3$$

is another curve parameterized by arclength with curvature κ and torsion τ, then there exists $\mathbf{w} \in \mathbb{R}^n$ and a rotation U of $\mathbb{R}^3$ such that

$$\bar{\mathbf{r}}(s) = U\mathbf{r}(s) + \mathbf{w}.$$

We will find an orthogonal transformation U with determinant 1. The fact that such a map on $\mathbb{R}^3$ is a rotation is an exercise in linear algebra which we will not do here.

To obtain U and $\mathbf{w}$, write $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$, $\bar{\mathbf{b}}$ for the Frenet frame for $\bar{\mathbf{r}}$ and set

$$U\,(x, y, z) \;=\; x\,\bar{\mathbf{t}}(s_0) \;+\; y\,\bar{\mathbf{n}}(s_0) \;+\; z\,\bar{\mathbf{n}}(s_0).$$

Then the linear map U is orthogonal and has determinant 1. Indeed, the linear map is orthogonal if

$$(U\mathbf{u} \cdot U\mathbf{v}) \;=\; \mathbf{u} \cdot \mathbf{v}.$$

This is equivalent to the fact that $U\mathbf{e}_1, U\mathbf{e}_2, U\mathbf{e}_3$ is orthonormal. Indeed, checking this on $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ we have

$$\begin{aligned}
(U\mathbf{u} \cdot U\mathbf{v}) \;&=\; (\,u_1\,\bar{\mathbf{t}}(s_0) \;+\; u_2\,\bar{\mathbf{n}}(s_0) \;+\; u_3\,\bar{\mathbf{b}}(s_0)\,) \;\cdot\; (\,v_1\,\bar{\mathbf{t}}(s_0) \;+\; v_2\,\bar{\mathbf{n}}(s_0) \;+\; v_3\,\bar{\mathbf{b}}(s_0)\,) \\
&=\; u_1v_1\bar{\mathbf{t}}(s_0)\cdot\bar{\mathbf{t}}(s_0) \;+\; u_2v_2\bar{\mathbf{n}}(s_0)\cdot\bar{\mathbf{n}}(s_0) \;+\; u_3v_3\bar{\mathbf{b}}(s_0)\cdot\bar{\mathbf{b}}(s_0) \\
&\;+\; (u_1v_2 + u_2v_1)\bar{\mathbf{t}}(s_0)\cdot\bar{\mathbf{n}}(s_0) \;+\; (u_1v_3 + u_3v_1)\bar{\mathbf{t}}(s_0)\cdot\bar{\mathbf{b}}(s_0) \;+\; (u_2v_3 + u_3v_2)\bar{\mathbf{n}}(s_0)\cdot\bar{\mathbf{b}}(s_0) \\
&=\; u_1v_1 + u_2v_2 + u_3v_3 \;=\; \mathbf{u}\cdot\mathbf{v}.
\end{aligned}$$

Furthermore,

$$\det U \;=\; \bar{\mathbf{t}}(s_0) \;\cdot\; \bar{\mathbf{n}}(s_0) \;\times\; \bar{\mathbf{b}}(s_0) \;=\; 1.$$

We set $\mathbf{w} = \bar{\mathbf{r}}(s_0)$.

$$T\mathbf{v} \;=\; U\mathbf{v} \;+\; \bar{\mathbf{r}}(s_0).$$

Then

$$T\mathbf{r}(s)$$

is a curve parameterized by arclength since

$$|(T\mathbf{r})'(s)| \;=\; |U(\mathbf{r}'(s))| \;=\; |U(\mathbf{t}(s))| \;=\; |\mathbf{t}(s)| \;=\; 1.$$

Furthermore,

$$T\mathbf{r}(s_0) = \bar{\mathbf{r}}(s_0)$$

and the Frenet frame at s_0 is

$$U(\mathbf{t}(s_0)) = \bar{\mathbf{t}}(s_0), \qquad \frac{(U\mathbf{t})'(s_0)}{|(U\mathbf{t})'(s_0)|} = \bar{\mathbf{n}}(s_0), \qquad \bar{\mathbf{t}}(s_0) \times \bar{\mathbf{n}}(s_0) = \bar{\mathbf{b}}(s_0).$$

By the uniqueness of solutions to ordinary differential equations,

$$T\mathbf{r} \;=\; \bar{\mathbf{r}}.$$

Regular Surfaces in $\mathbb{R}^3$

We would like to define a *regular surface* in $\mathbb{R}^3$ so let us first of all decide what that should be. The unit sphere and a 'smooth' torus in $\mathbb{R}^3$ should certainly qualify. We also want to be able to take non-closed surfaces like the hemisphere and disc, although we don't want to include their 'edges' at this stage. We are going to consider surfaces to be subsets of $\mathbb{R}^3$, unlike curves which are maps of an interval into $\mathbb{R}^3$. The definition of a surface will be more analogous to the trace of a parameterized curve than to a parameterized curve itself, although we will not allow the surface to 'cross itself'

If (X, d) is a metric space, $u \in X$, and $\varepsilon > 0$, write $B_\varepsilon(u) = \{v \in X : d(v, u) < \varepsilon\}$.
Definition $U \subset X$ is *open* if for each $u \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(u) \subseteq U$.
Examples. 1. In any metric space (U, d), $B_\rho(v)$ is open, since for $u \in B_\rho(v)$, if $w \in B_{\rho - d(u,v)}(u)$ then

$$d(w, v) \leq d(u, v) + d(u, w) < \rho$$

so $w \in B_\rho(v)$.
2. The spaces X and $\emptyset$ are open.
3. In $\mathbb{R}^n$ the set $\{v : |u - v| \leq \varepsilon\}$ is not open.
The concept of an open set in $\mathbb{R}^n$ is useful if we want to define differentiable functions on the set, since you can approach a point in an open set from any direction. The local structure of an open set in $\mathbb{R}^n$ is simple because around any point it just looks like $\mathbb{R}^n$.
Definition. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$ there exists an open set $U \subset \mathbb{R}^2$ and an open set $V \subset \mathbb{R}^3$ containing p and a map $\mathbf{r} : U \to V \cap S$ such that
1. $\mathbf{r}$ is smooth.
2. $\mathbf{r}$ is a homeomorphism of U onto $V \cap S$.
3. For each $\mathbf{q} \in U$, the differential $d\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The map $\mathbf{r}$ is called a *parameterization* or a *system of local coordinates* around p.

We illustrate these properties with an example. We will show that the unit sphere $x^2 + y^2 + z^2 = 1$ is a regular surface. We start by defining a parameterization $\mathbf{r}$ for points in the upper hemisphere. Set U to be the unit disc $u^2 + v^2 < 1$

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \qquad x = u, \quad y = v, \quad z = \sqrt{1 - u^2 - v^2}.$$

1. This is smooth because the partial derivatives of all orders

$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial^2 x}{\partial u \partial v}, \ldots \qquad , \frac{\partial y}{\partial u}, \ldots\ldots \qquad , \frac{\partial z}{\partial u} \ldots\ldots$$

exist and are continuous.
2. **Definitions.** If (X, d_X) and (Y, d_Y) are metric spaces then the map $f : X \to Y$ is *continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(u), f(v)) < \varepsilon$ when $d(u, v) < \delta$. Equivalently, whenever $u_i \to u$ in X as $i \to \infty$ then $f(u_i) \to f(u)$ in Y.

f is a *homeomorphism* if f is a bijection (i.e. it is one-to-one and onto), f is continuous, and f^{-1} is continuous. This says that when identified by the map f, a sequence in X converges exactly when the corresponding sequence in Y converges.

The parameterization (*) for the sphere is continuous. This was already checked in Step 1. It is a bijection because we can write down its inverse:

$$\mathbf{r}^{-1} : (x, y, z) \;=\; (x, y).$$

This is continuous on $\mathbb{R}^3$, and hence it is continuous when restricted to the upper hemisphere. Condition 2 ensures that the surface doesn't loop around and meet itself.
3. $d\mathbf{r}$ has matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}.$$

The linear map defined by this matrix is clearly one-to-one. Indeed, its columns are linearly independent. In this case it is obvious that one column is not a multiple of the other. In general this can be determined by calculating the cross product of the columns.
Remark. $d\mathbf{r}(\mathbf{u})$ is the linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$\mathbf{r}(\mathbf{u} + \delta\mathbf{u}) \;=\; \mathbf{r}(\mathbf{u}) \;+\; d\mathbf{r}(\mathbf{u})\,\delta\mathbf{u} \;+\; O(|\delta\mathbf{u}|^2).$$

Using matrix multiplication,

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \;=\; \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}.$$

We have now found a parameterization for the upper hemisphere. We must however give enough parameterizations to cover all points of the sphere.