LECTURE 5: REGULAR SURFACES.

October 3, 2001

Let's review the derivative of a map. Suppose that U is open in $\mathbb{R}^2$ and $\mathbf{r} : U \to \mathbb{R}^3$ is smooth . What does it mean for $\mathbf{r}$ to be continuously differentiable at $p \in U$? It means that the partial derivatives

$$\frac{\partial \mathbf{r}}{\partial u}(u,v) = \lim_{h \to 0} \frac{\mathbf{r}(u+h,v) - \mathbf{r}(u,v)}{h}, \qquad \frac{\partial \mathbf{r}}{\partial v}(u,v) = \lim_{h \to 0} \frac{\mathbf{r}(u,v+h) - \mathbf{r}(u,v)}{h}$$

exist and are continuous. In this case one can prove that the *chain rule* holds: If $\mathbf{u}(t) = (u(t), v(t))$ is a smooth curve with $(u(0), v(0)) = \mathbf{u}_0$, $(u'(0), v'(0)) = (a,b)$ then

$$\begin{aligned} \left.\frac{d\mathbf{r}(\mathbf{u}(t))}{dt}\right|_{t=0} &= \left.\frac{\partial \mathbf{r}}{\partial u}\right|_{\mathbf{u}_0} \left.\frac{du}{dt}\right|_0 + \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{\mathbf{u}_0} \left.\frac{dv}{dt}\right|_0 \\ &= \left.\frac{\partial \mathbf{r}}{\partial u}\right|_{\mathbf{u}_0} a + \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{\mathbf{u}_0} b. \end{aligned}$$

This is called the *directional derivative of* $\mathbf{r}$ *in the direction* (a,b). Writing vectors as columns for a moment, the linear map

$$d\mathbf{r}_{\mathbf{u}_0} : \begin{pmatrix} a \\ b \end{pmatrix} \to \left.\frac{\partial \mathbf{r}}{\partial u}\right|_{\mathbf{u}_0} a + \left.\frac{\partial \mathbf{r}}{\partial v}\right|_{\mathbf{u}_0} b = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

is the *derivative* or *differential* of $\mathbf{r}$ at $\mathbf{u}_0$.

We recap what happened last time.

Definition. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$ there exists an open set $U \subset \mathbb{R}^2$ and an open set $V \subset \mathbb{R}^3$ containing p and a map $\mathbf{r} : U \to V \cap S$ such that

1. $\mathbf{r}$ is smooth.
2. $\mathbf{r}$ is a homeomorphism of U onto $V \cap S$.
3. For each $\mathbf{q} \in U$, the differential $d\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The map $\mathbf{r}$ is called a *parameterization* around p.

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Conditions 1 and 3 ensure that $\mathbf{r}$ is approximated by a one-to-one linear map close to each point, so S is approximated by a tangent plane.

Example. $\mathbf{r}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$ for $\rho \in \mathbb{R}$ and $0 < \theta < 2\pi$. $d\mathbf{r}$ has matrix

$$\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \\ 1 & 0 \end{pmatrix}.$$

This is one-to-one unless the columns are dependent. We can check this by computing the cross product

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos\theta & \sin\theta & 1 \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = (-\rho\cos\theta, -\rho\sin\theta, \rho).$$

This vanishes if and only if $\rho = 0$. The image of $\mathbf{r}$ is a cone and $\rho = 0$ corresponds to the cone point.
Condition 2 ensures that $V \cap S$ doesn't bend around and meet itself.

Last time we showed that

$$\mathbf{r}(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$$

is a parameterization from the unit disc to the upper hemisphere. In order that every point of the sphere should be covered by a parameterization, we can take six parameterizations from the unit disc in $\mathbb{R}^2$ to the sphere:

$$\begin{aligned} \mathbf{r}_1(x, y) &= (x, y, \sqrt{1 - x^2 - y^2}) \\ \mathbf{r}_2(x, y) &= (x, y, -\sqrt{1 - x^2 - y^2}) \\ \mathbf{r}_3(y, z) &= (\sqrt{1 - y^2 - z^2}, y, z) \\ \mathbf{r}_4(y, z) &= (-\sqrt{1 - y^2 - z^2}, y, z) \\ \mathbf{r}_5(x, z) &= (x, \sqrt{1 - x^2 - z^2}, z) \\ \mathbf{r}_6(x, z) &= (x, -\sqrt{1 - x^2 - z^2}, z). \end{aligned}$$

Lemma. The graph of a smooth function over an open set is a regular surface. Indeed, if $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is smooth, set

$$\mathbf{r}(x, y) = (x, y, f(x, y)).$$

Then $S = \mathbf{r}(U)$ is a regular surface.

Proof. 1. $\mathbf{r}$ is smooth since f is smooth.
2. $\mathbf{r}$ is one-to-one and onto and the inverse is given by

$$\mathbf{r}^{-1}(x,y,z) \;\to\; (x,y)$$

which is continuous on S since it is continuous on all of $\mathbb{R}^3$.
3. $d\mathbf{r}_q$ has matrix

$$\begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}.$$

This is clearly one-to-one.

Definition. If (X,d) is a metric space and $p \in X$ then an *open neighborhood of p is an open set containing p. It can be shown that if $S \subset \mathbb{R}^3$, an open neighborhood of p in S has the form $V \cap S$ where V is an open set in $\mathbb{R}^3$.*

Theorem. *If S is a regular surface and $p \in S$ then there exists an open neighborhood V of p in S such that V has one of the forms $z = f(x,y)$, $y = g(x,z)$, $z = h(x,y)$ for smooth f, g, h.*

Which form does it have? Take a parameterization around p,

$$\mathbf{r} : (u,v) \;\to\; (x(u,v), y(u,v), z(u,v)).$$

Follow this with the projection onto a coordinate plane e.g.

$$\pi_{xy} : (x,y,z) \;\to\; (x,y).$$

Then

$$\pi\mathbf{r} : (u,v) \;\to\; (x(u,v), y(u,v)).$$

If the derivative at p is one-to-one then S, that is the Jacobian does not vanish,

$$\frac{\partial(x,y)}{\partial(u,v)} \;\neq\; 0,$$

then S can be written as $z = f(x,y)$ close to p. The proof is a consequence of the inverse function theorem. This says that if a smooth map F from an open set of $\mathbb{R}^n$ to $\mathbb{R}^n$ has dF_q one-to-one, then F is invertible on a neighborhood of q and the inverse is smooth.

In our case, let G be an inverse to $\pi_{xy}\mathbf{r}$ which is defined on a neighborhood of πp. Then $\pi\_xybrG = I$, the identity. Hence $\mathbf{r}G$ gives S as a graph close to πp.

Definition. *Now suppose* $U \subset \mathbb{R}^3$ *is open and* $F : U \to \mathbb{R}$ *is smooth. We say that* $q \in U$ *is a critical point of* F *if the gradient* dF_q *vanishes. Note that* dF *has 'matrix'*

$$\left(\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \quad \frac{\partial F}{\partial z} \right).$$

We say that $a \in F(U)$ *is a regular value of* F *if there are no critical points* (x, y, z) *of* F *with* $F(x, y, z) = a$.
The most useful result for checking whether a set is a regular surface is the following
Theorem. *Suppose that* $U \subset \mathbb{R}^3$ *is open and* $F : U \to \mathbb{R}$ *is smooth. Suppose that* a *is a regular value of* F*. Then*

$$F^{-1}(a) \;=\; \{\mathbf{r} : F(\mathbf{r}) = a\}$$

is a regular surface.
This is also proved using the Open Mapping Theorem.
Examples.

$$F(x, y, z) = x^2 + y^2 + z^2, \qquad dF = (2x, 2y, 2z) = (0, 0, 0) \Leftrightarrow (x, y, z) = (0, 0, 0).$$

Hence 1 *is a regular value of* F *and the unit sphere* $x^2 + y^2 + z^2 = 1$ *is a regular surface.*

$$F(x, y, z) = x^2 + y^2 - z^2, \qquad dF = (2x, 2y, -2z) = (0, 0, 0) \Leftrightarrow (x, y, z) = (0, 0, 0).$$

The only value of F *which is not regular is* 0*. The set* $F = 0$ *is a double cone, which is indeed not a regular surface. The problem is the point* $(0, 0, 0)$*. If we consider the open cone* $F = 0$ *with* $z > 0$*, this is a regular surface. Indeed,* 0 *is a regular value of* F *on* $z > 0$*.*
The one sheet hyperboloid corresponds to $F = a > 0$ *and the two sheet hyperboloid corresponds to* $F = a < 0$*. These are regular surfaces. (Note that a regular surface need not be connected.)*

Assume $c > 1$*. Then the following equations describe a torus.*

$$F = (\sqrt{x^2 + y^2} - c)^2 + z^2, \qquad dF = \left(\frac{2x(\sqrt{x^2 + y^2} - c)}{\sqrt{x^2 + y^2}}, \frac{2y(\sqrt{x^2 + y^2} - c)}{\sqrt{x^2 + y^2}}, 2z \right).$$

Assuming $c \neq 0$*, this is singular if* $(x, y) = (0, 0)$*. This cannot occur if* $c > 1$*.* $dF = 0$ *if* $x^2 + y^2 = c$ *and* $z = 0$*, which corresponds to* $F = 0$ *and the torus degenerates to a circle.*

Lemma. *Suppose that* S *is a regular surface,* U *is an open subset of* $\mathbb{R}^2$*, and* $\mathbf{r} : U \to S$ *is a smooth one-to-one function with one-to-one differential* $d\mathbf{r}_q$ *at each point* $q \in U$*. Then* $\mathbf{r}$ *is a parameterization. (i.e. you don't need to check that* $\mathbf{r}^{-1}$ *is continuous.*

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Homework. *Show that if a map* T *from* 3*-space to itself preserves distance, that is* $|T\mathbf{u} - T\mathbf{v}| = |\mathbf{u} - \mathbf{v}|$ *for all* $\mathbf{u}$*, then* T *has the form* $T\mathbf{u} = O\mathbf{u} + \mathbf{w}$*, where* $\mathbf{w}$ *is a* 3*-vector and* O *is a linear orthogonal transformation of* 3*-space.*
§2-2, #1, 2, 3, 4, 5, 6, 15, 16, 17.