LECTURE 6: THE INVERSE FUNCTION THEOREM.

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Today's Quiz:
1. If $\mathbf{r}(s)$ is a regular curve whose curvature vanishes everywhere, what can you say about it?
2. If $\mathbf{r}(s)$ is a regular curve whose torsion vanishes everywhere, what can you say about it?
3. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear map

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

what is $dF_{(x,y,z)}$?
4. If the linear map F above is one-to-one then what can you say about the image $F(\mathbb{R}^3)$?

The Inverse Function Theorem. Suppose U is open in $\mathbb{R}^n$ and $p \in U$. If $F : U \to \mathbb{R}^n$ is smooth and dF_p is invertible then there exist open neighborhoods $V \subset U$ of p and W of $F(p)$ such that F is a diffeomorphism from V to W, that is $F^{-1} : W \to V$ exists and is smooth.

Sketch of the proof. If we can show that $F : V \to W$ is a homeomorphism, and if dF_q is invertible for $q \in V$ then the fact that F^{-1} is smooth follows. Indeed, if $F(u) = x$ and $F(v) = y$ then since F is differentiable at v,

$$\frac{|F(u) - F(v) - dF_v(u - v)|}{|u - v|} \to 0, \qquad u \to v.$$

This becomes

$$\frac{|x - y - dF_v(F(x) - F(y))|}{|F(x) - F(y)|} = \frac{|dF_v(F(x) - F(y) - dF_v^{-1}(x - y))|}{|x - y|} \frac{|x - y|}{|F(x) - F(y)|} \to 0.$$

With a little more work, this shows that F^{-1} is differentiable with $d(F^{-1})_y = (dF_v)^{-1}$, and this can be used to show that F^{-1} is smooth.

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Now we show that locally F is a homeomorphism. For y close to $F(p)$ we need to solve $F(x) = y = 0$, that is $F(x) - y = 0$.
Newton's method: If $f : [a, b]R \to \mathbb{R}$ is continuous, we can try to find a solution of $f(x) = 0$ by Newton iteration:

$$x_{i+1} \;=\; x_i \;-\; \frac{f(x_i)}{f'(x_i)}.$$

We can put this in the setting of the contraction mapping theorem by setting

$$Tx \;=\; x \;-\; \frac{f(x)}{f'(x)}.$$

Then

$$Tx = x \qquad \Leftrightarrow \qquad f(x) = 0.$$

If we can show that f is a contraction on $[a, b]$ then we get a unique solution.
Back to the open mapping theorem: To solve $F(x) - y$ we set

$$Tx \;=\; x \;-\; (dF_p)^{-1}(F(x) - y).$$

Notice that we have fixed the point p where we evaluate $(dF_p)^{-1}$ unlike in Newton's method. We assume that dF_p is the identity, and $p = q = 0$. It is easy to reduce to this case by composing with a linear map. Then we set

$$Tx \;=\; x - F(x) + y.$$

Clearly

$$F(x) = y \qquad \Leftrightarrow \qquad Tx = x.$$

The object is to show that when $|y| < \delta$ then T is a contraction of $|x| \leq \varepsilon$, for some small ε and δ. The contraction mapping theorem then gives the solution.

Problem. Show that $xyz = 1$ is a regular surface.
Set $f = xyz$. Then $df = (xy, yz, xz)$. If this equals $(0, 0, 0)$ then $xyz = 0$. Hence 1 is a regular value of f and $f = 1$ is a regular surface.

Theorem. If a is a regular value of the smooth function $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^3$ is open, then $S = f^{-1}(a)$ is a regular surface.
Proof. Let $p \in F^{-1}(a)$. We must parameterize F around p. Now since p is not a critical point of f, one of $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ is non-vanishing at p. Assume $\partial f/\partial z \neq 0$ at p. Consider the map $F : U \to \mathbb{R}^3$ given by

$$F(x, y, z) \;=\; (x, y, f(x, y, z)).$$

Then the derivative dF_p has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}.$$

The determinant is $\partial f(p)/\partial z \neq 0$, so dF_p is invertible. By the inverse function theorem there exist V and W neighborhoods of p and $F(p)$ such that F is a diffeomorphism from V to W. Let G be the inverse. Then

$$G(x, y, f) = (x, y, g(x, y, f)),$$

but

$$G(x, y, f(x, y, z)) = (x, y, z) \qquad\qquad g(x, y, f(x, y, z)) = z.$$

Set

$$\mathbf{r}(x, y) = G(x, y, a)$$

on $\{(x, y) : (x, y, a) \in W$. Then $\mathbf{r}$ is a parameterization of S around p. Indeed, it is the graph of the smooth function $g(x, y, a)$.

Lemma. Suppose that $\mathbf{r} : U \to S$ is a map from the open set $U \subset \mathbb{R}^2$ to the regular surface S such that

1. $\mathbf{r}$ is smooth.
2'. $\mathbf{r}$ is one-to-one.
3. $d\mathbf{r}_q$ is one-to-one for each $q \in U$.

Then $\mathbf{r}$ is a parameterization of S - i.e. the inverse of $\mathbf{r}$ is continuous.
Example. Show that usual spherical coordinates

$$(\theta, \phi) \;\to\; (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \qquad\qquad 0 < \phi < \pi, \quad 0 < \theta < 2\pi,$$

give a parameterization of the sphere.
The map is continuous. It is also one-to-one as one can check. Indeed, suppose

$$(\sin\phi_1\cos\theta_1, \sin\phi_1\sin\theta_1, \cos\phi_1) \;=\; (\sin\phi_2\cos\theta_2, \sin\phi_2\sin\theta_2, \cos\phi_2)$$

since $\cos\phi_1 = \cos\phi_2$, then $\phi_1 = \phi_2$, so

$$(\cos\theta_1, \sin\theta_1) \;=\; (\cos\theta_2, \sin\theta_2)$$

so $\theta_1 = \theta_2$. The differential has matrix

$$\begin{pmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \sin\phi\cos\theta & \cos\phi\sin\theta \\ 0 & -\sin\phi \end{pmatrix}$$

and the vector product of the columns is

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix} = (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi).$$

If this is zero then $\cos\phi\sin\phi = 0 \Rightarrow \phi = \pi/2$ and $\sin\theta = \cos\theta = 0$ which is a contradiction.