LECTURE 7: SMOOTH FUNCTIONS BETWEEN REGULAR SURFACES.

October 10, 2001

Today's Quiz:
1. Suppose $f : X \to Y$ is a continuous map between metric spaces (X, d_X) and (Y, d_Y). That is for each $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$.
   (a). If $x_i \to x$ as $i \to \infty$, what can you say about $f(x_i)$?
   (b). If V is open in Y, what can you say about $f^{-1}(V)$?
   (c). If U is open in X what can you say about $f(U)$?
2. What is the maximum area of the plane which can be enclosed by a curve of length 1?

We will need 1(b). The answer is that $U = f^{-1}(V)$ is open if V is open. Indeed, if $u \in U$ then pick $\varepsilon > 0$ such that $\{v : d_V(v, f(u)) < \varepsilon\} \subset V$. Then pick $\delta > 0$ such that $d_U(u', u) < \delta$ then $d_V(f(u'), f(u)) < \varepsilon$ so $f(u) \in V$. Hence $\{u' : d(u', u) < \varepsilon\} \subset U$ so U is open.

Smooth functions on regular surfaces.

We know what it means for a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ to be smooth, but what about a function defined on a surface?
Definition. Suppose that $f : S \to \mathbb{R}$ is a function on the regular surface S. Then f is *smooth* if for every $p \in S$, there exists a parameterization $\mathbf{r}$ of S around p such that the composition $f \circ \mathbf{r}$ is smooth.

Useful Fact. If U is open in $\mathbb{R}^3$ and $S \subset U$ is a regular surface and $f : U \to \mathbb{R}$ is smooth then the restriction $f : S \to \mathbb{R}$ is smooth. This is clear because the composition $f \circ \mathbf{r}$ is a composition of smooth functions hence smooth.

Question. James asked where we used the fact that the surface is regular. We used this fact to some extent in asserting that the parameterization is smooth, but we can of course find smooth "parameterizations" of surfaces which are not regular - like the cone $z^2 = x^2 + y^2$. (Such a parameterization would have a derivative which is not one-to-one at the cone point.) It is indeed possible to put a "smooth structure" on the cone (i.e. to define a class of smooth functions on the cone) by choosing a "parameterization" $\mathbf{r}$ of the cone and asserting that the smooth functions on the cone are to be those functions f such that $f \circ \mathbf{r}$ is smooth. However, this smooth structure will not be canonical. The smooth structure we defined on a regular surface is the unique one such that the smooth functions on the cone are precisely the restrictions of smooth functions on an open set in $\mathbb{R}^3$ to the cone.

Example. The function

$$f = \frac{1}{(x-1)^2 + (y-1)^2 + (z-1)^2}$$

is smooth on the unit sphere $x^2 + y^2 + z^2 = 1$. Indeed, f is smooth on $\mathbb{R}^3 \setminus (1,1,1)$ and this is an open set containing the sphere.

Lemma. (You can fatten up a parameterization.) If $\mathbf{r} : U \to S$ is a parameterization from the open set $U \subset \mathbb{R}^2$ to the regular surface S and $q \in U$, then there exists an open neighborhood $V \subset U$ of q, $\varepsilon > 0$ and an open neighborhood W of $p = \mathbf{r}(q)$ in $\mathbb{R}^3$ and a map $\mathbf{R} : V \times (-\varepsilon, \varepsilon) \to W$ such that

1. $\mathbf{R}$ is a diffeomorphism.
2. $\mathbf{R}(u, v, 0) = \mathbf{r}(u, v)$, for all $(u, v) \in V$.
3. $S \cap W = \mathbf{r}(V)$.

Proof. Since $\mathbf{r}$ is a parameterization, writing $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, we have that one of the Jacobians

$$\frac{\partial(x,y)}{\partial(u,v)}, \qquad \frac{\partial(x,z)}{\partial(u,v)}, \qquad \frac{\partial(y,z)}{\partial(u,v)}$$

is non-zero. Suppose without loss of generality that at $\mathbf{q}$,

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0.$$

Then define

$$\mathbf{R}(u, v, t) \ = \ \mathbf{r}(u, v) \ + \ t\mathbf{e}_3, \qquad\qquad (u, v) \in U.$$

Then $\mathbf{R}$ is smooth on $U \times \mathbb{R}$. Furthermore $d\mathbf{R}$ has matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{pmatrix}$$

which is invertible at $(u_0, v_0, 0)$ where $\mathbf{q} = (u_0, v_0)$. By the inverse function theorem there exists an open neighborhood V_0 of $(u_0, v_0, 0)$ in $\mathbb{R}^3$ and an open neighborhood W_0 of $\mathbf{p}$ in $\mathbb{R}^3$ such that $\mathbf{R}$ is a diffeomorphism from V_0 to W_0. Now we already proved that there is a neighborhood W_1 of $\mathbf{p}$ in $\mathbb{R}^3$ such that $S \cap W_1$ is a graph $z = f(x, y)$. Then $\mathbf{R}$ is a diffeomorphism from $V_1 = \mathbf{R}^{-1}(W_0 \cap W_1)$ onto $W_0 \cap W_1$. Since V_1 is open and contains $(u_0, v_0, 0)$, it contains a ball about $(u_0, v_0, 0)$ of some positive radius $\sqrt{2}\varepsilon$. But then it contains $V_2 = V \times (-\varepsilon, \varepsilon)$ where V is a ball in $\mathbb{R}^2$ with center (u_0, v_0) and radius ε. Then $\mathbf{R}$ is a diffeomorphism from V_2 to $\mathbf{R}(V_2) = W$. Furthermore $\mathbf{R}(u, v, 0) = \mathbf{r}(u, v)$ by definition. Now $\mathbf{r}(V) \subset S \cap W$, and in fact they must be equal, for $\mathbf{r}(u, v) = \mathbf{R}(u, v, 0) \in S$ and so if $0 < |t| < \varepsilon$, $\mathbf{R}(u, v, t) = \mathbf{R}(u, v, 0) + t\mathbf{e}_3$ cannot also be in S because $S \cap W$ is a graph over the xy plane.