LECTURE 8: DIFFEOMORPHISMS BETWEEN REGULAR SURFACES.

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Lemma. (You can fatten up a parameterization.) If $\mathbf{r} : U \to S$ is a parameterization from the open set $U \subset \mathbb{R}^2$ to the regular surface S and $q \in U$, then there exists an open neighborhood $V \subset U$ of q, $\varepsilon > 0$ and an open neighborhood W of $p = \mathbf{r}(q)$ in $\mathbb{R}^3$ and a map $\mathbf{R} : V \times (-\varepsilon, \varepsilon) \to W$ such that

1. $\mathbf{R}$ is a diffeomorphism.
2. $\mathbf{R}(u, v, 0) = \mathbf{r}(u, v)$, for all $(u, v) \in V$.
3. $S \cap W = \mathbf{r}(V)$.

Theorem. (Change of parameters is smooth.) If U_1, U_2 are open in $\mathbb{R}^2$ and

$$\mathbf{r}_1 : U_1 \to S$$
$$\mathbf{r}_2 : U_2 \to S$$

Are two parameterizations of S, then $\mathbf{r}_2^{-1}\mathbf{r}_1$ is smooth on $\mathbf{r}_1^{-1}(\mathbf{r}_2 U_2)$.

Proof. Let $\mathbf{r}_1(\mathbf{q}_1) = \mathbf{p} = \mathbf{r}_2(\mathbf{q}_2)$. Pick $V_2 \subset U_2$ an open neighborhood of $\mathbf{q}_2$, and $\varepsilon > 0$ and let W_1 be an open neighborhood of p in $\mathbb{R}^3$ such that there exist a diffeomorphism

$$\mathbf{R}_2 : V_2 \times (-\varepsilon, \varepsilon) \to W_2$$

extending $\mathbf{r}_2$ in the sense of the fattening lemma, that is $\mathbf{R}_2(u, v, 0) = \mathbf{r}_2(u, v)$, for all $(u, v) \in V_2$, and $S \cap W_2 = \mathbf{r}_2(V_2)$. Then $\mathbf{r}_2^{-1}\mathbf{r}_1 = \mathbf{R}_2^{-1}\mathbf{r}_1$ is smooth on $\mathbf{r}^{-1}W_2$.

Recall. $f : S \to \mathbb{R}$ is smooth if for each $\mathbf{p} \in S$, $f \circ \mathbf{r}$ is smooth for some parameterization $\mathbf{r}$ of S around $\mathbf{p}$. We now see that this is equivalent to: the fact that $f \circ \mathbf{r}$ is smooth for every parameterization $\mathbf{r}$.

Definition. A map $F : S_1 \to S_2$ between the regular surfaces S_1 and S_2 is smooth if F is continuous, and for every $\mathbf{p}_1 \in S_1$, there exists a pair of parameterizations $\mathbf{r}_1 : U_1 \to S_1$ around $\mathbf{p}_1$ and $\mathbf{r}_2 : U_2 \to S_2$ around $\mathbf{p}_2 = F(\mathbf{p}_1)$ with $F\mathbf{r}_1(U_1) \subset U_2$, such that

$$\mathbf{r}_2^{-1} F \mathbf{r}_1$$

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is smooth on U_1.
Equivalently, $F : S_1 \to S_2$ is smooth if F is continuous, and for every pair of parameterizations $\mathbf{r}_1 : U_1 \to S_1$ and $\mathbf{r}_2 : U_2 \to S_2$,

$$\mathbf{r}_2^{-1} F \mathbf{r}_1$$

is smooth on $\mathbf{r}_1^{-1} F^{-1} \mathbf{r}_2 U_2$.

Definition. The map $F : S_1 \to S_2$ between the regular surfaces S_1 and S_2 is a *diffeomorphism* if it is smooth and has a smooth inverse

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If there exists a diffeomorphism $F : S_1 \to S_2$ then S_1 and S_2 are *diffeomorphic.*

Example. Any parameterization $\mathbf{r} : U \to S$ of the regular surface S is a diffeomorphism onto its image. Indeed, it is continuous by the definition of parameterization. We can take the parameterization of U to be the identity I and the parameterization of $\mathbf{r}(U)$ to be $\mathbf{r}$. Then $\mathbf{r}$ is smooth because the composition $\mathbf{r}^{-1}\mathbf{r}I = I$ which is smooth. The inverse $\mathbf{r}^{-1}$ from $\mathbf{r}(U)$ to U is also smooth by the same argument.

Theorem. If $V_1 \subset \mathbb{R}^3$ is open and $F : V_1 \to \mathbb{R}^3$ is smooth and S_1, S_2 are regular surfaces with $S_1 \subset V_1$ and $F(S_1) \subset S_2$, then F is smooth from S_1 to S_2.

Example. Show that the map

$$F : (x, y, z) \ \to \ \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z \right)$$

is a diffeomorphism between the one sheet hyperboloid $x^2 + y^2 = z^2 + 1$ and the cylinder $x^2 + y^2 = 1$.
Solution. F as defined is smooth on $V = \mathbb{R}^3 \setminus \{(0, 0, z)\}$. Since the hyperboloid is contained in V and F maps it onto the cylinder, F is smooth from the hyperboloid to the cylinder. Set

$$G(x, y, z) \ = \ \left(\sqrt{z^2 + 1}x, \sqrt{z^2 + 1}y, z \right).$$

Then G is smooth on $\mathbb{R}^3$ and restricted to the cylinder it is the inverse of F. Hence F is a diffeomorphism.

Proof of the Theorem. For $\mathbf{p} \in S_1$, take the pair of parameterizations $\mathbf{r}_1 : U_1 \to S_1$ about $\mathbf{p}$ and $\mathbf{r}_2 : U_2 \to S_2$ with $F\mathbf{r}_1(U_1) \subset \mathbf{r}_2(U_1)$. Now fatten up the parameterization $\mathbf{r}_2$, i.e. let $\mathbf{R}_2 : V_2 \times (-\varepsilon, \varepsilon) \to W_2$ be such that $\mathbf{r}_2 F(\mathbf{p}) \in V_2$, $\mathbf{R}_2(\mathbf{q}, 0) = \mathbf{r}_2(\mathbf{q})$ and $W_2 \cap S_2 = \mathbf{r}_2(V_2)$. Then $\mathbf{r}_2^{-1} F \mathbf{r}_1 = \mathbf{R}_2^{-1} F \mathbf{r}_1$ is smooth on $\mathbf{r}_1^{-1} F^{-1} \mathbf{r}_2 V_2$.

Example. The map $(x, y, z) \to (ax, by, cz)$ is a diffeomorphism from the sphere $x^2 + y^2 + z^2 = 1$ onto the ellipse $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Indeed, the map is smooth on $\mathbb{R}^3$ and hence smooth on the sphere. Furthermore, it does map the sphere into the ellipse, and an inverse is given by $(x, y, z) \to (x/a, y/b, z/c)$ which is smooth on $\mathbb{R}^3$ hence on the ellipse.

Definition. A *regular parameterized surface* is a smooth map $\mathbf{r} : U \to \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ is open, such that $d\mathbf{r}_\mathbf{q}$ is one-to-one for all $\mathbf{q} \in U$.

Example.

$$\mathbf{r}(u, v) \;=\; (\cos u, \sin 2u, v)$$

is a regular parameterized surface. In particular $d\mathbf{r}$ has matrix

$$\begin{pmatrix} -\sin u & 0 \\ 2\cos 2u & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\sin u & 2\cos 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} \;=\; (2\cos 2u, \sin u, 0)$$

which equals zero if $\sin u = 0$ and $\cos 2u = \cos^2 u - \sin^2 u = 0$, so $\cos u = 0$ which is a contradiction. Hence $d\mathbf{r}_\mathbf{q}$ is one-to-one for each $\mathbf{q}$ and $\mathbf{r}$ is a regular parameterized surface. Notice that $\mathbf{r}$ is not one-to-one and the image is not a regular surface.

Theorem. If $\mathbf{r} : U \to \mathbb{R}^3$ is a regular parameterized surface and $\mathbf{q} \in U$ then there exists a neighborhood V of $\mathbf{q}$ such that $\mathbf{r}(V)$ is a regular surface.

Proof. We show that we can find V so that $\mathbf{r}(V)$ is a smooth graph over a coordinate plane. This is exactly the proof we gave before, except for one point. First note that one of the Jacobians

$$\frac{\partial(x, y)}{\partial(u, v)}, \qquad \frac{\partial(x, z)}{\partial(u, v)}, \qquad \frac{\partial(y, z)}{\partial(u, v)}$$

does not vanish at $\mathbf{q}$. WLOG

$$\frac{\partial(x, z)}{\partial(u, v)} \neq 0$$

at $\mathbf{q}$. Then writing π for the projection onto the xz plane, $d(\pi\mathbf{q}) \neq 0$ at $\mathbf{q}$. By the inverse function theorem there exists a neighborhood $V \subset U$ such that $\pi\mathbf{q}$ is a diffeomorphism from V to $\pi\mathbf{q}V$. But then defining

$$F(x, z) \;=\; \mathbf{q} \circ (\pi \circ \mathbf{q})^{-1}$$

we have $F \circ \pi$ is the identity on $\pi\mathbf{q}V$, and F gives $\mathbf{r}(V)$ as a graph over the xz plane.

Lemma. Suppose $\mathbf{r} : U \to S$ is smooth where $U \subset \mathbb{R}^2$ is open and

1. $\mathbf{r}$ is smooth.

2′. $\mathbf{r}$ is one-to-one.

3. $d\mathbf{r}_{\mathbf{q}}$ is one-to-one for each $\mathbf{q} \in S$.

Then $\mathbf{r}$ is a parameterization.

Proof. For $\mathbf{p} \in \mathbf{r}(U)$, we must show that $\mathbf{r}^{-1}$ is continuous at $\mathbf{p}$. Take an open neighborhood W of $\mathbf{p}$ in S such that locally W is the graph over one of the coordinate planes, that is for π the projection onto, say, the xz plane, W is the image of

$$G : V_0 \to \mathbb{R}^3$$

where G is smooth with $G\pi$ equal to the identity. Then for $\mathbf{q} = \mathbf{t}^{-1}\mathbf{p}$, we have $\mathbf{r} = G\pi\mathbf{r}$ so $d\mathbf{r} = dF \circ d\pi \circ d\mathbf{r}$ is one-to-one and $d(\pi \circ \mathbf{r})$ is one-to-one. Following the previous proof we find that $\pi \circ \mathbf{r}$ is locally invertible and hence $\mathbf{r}$ is.