

**LECTURE 8: THE TANGENT SPACE  
AND THE DIFFERENTIAL OF A MAP.**

October 22, 2001

**Today's Quiz:**

1. What geometric information do you need to describe a plane in  $\mathbb{R}^3$ ? What equations can be used to describe a plane?
2. Solve

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

How do you compute the vector  $\mathbf{u}$ ?

3. Suppose  $\mathbf{u}$  and  $\mathbf{w}$  span a plane containing the vector  $\mathbf{w}$ , so  $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$ . Give a formula to compute  $\lambda, \mu$ .

**Definition.** If  $S$  is a regular surface, and  $\mathbf{p} \in S$ , a *tangent vector* to  $S$  at  $\mathbf{p}$  is a vector of the form  $\alpha'(0)$  where  $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  is any smooth parameterized curve whose image lies in  $S$  with  $\alpha(0) = \mathbf{p}$ .

**Theorem.** If  $\mathbf{r} : U \rightarrow S$  is a parameterization of  $S$  with  $\mathbf{r}(\mathbf{q}) = \mathbf{p}$ , the set of tangent vectors to  $S$  at  $\mathbf{p}$  is  $d\mathbf{r}_{\mathbf{q}}(\mathbb{R}^2)$ . In particular, the set of tangent vectors is a plane and  $d\mathbf{r}_{\mathbf{q}}(\mathbb{R}^2)$  is independent of the parameterization.

Proof. First we note that  $d\mathbf{r}_{\mathbf{q}}(\mathbb{R}^2)$  is contained in the set of tangent vectors. Indeed, if  $(\lambda, \mu) \in \mathbb{R}^2$  then  $\alpha(t) = \mathbf{r}(\mathbf{q} + (t\lambda, t\mu))$  is a curve in  $S$  with  $\alpha(0) = \mathbf{p}$  and

$$\alpha'(0) = d\mathbf{r}_{\mathbf{q}} \frac{d(\mathbf{q} + (t\lambda, t\mu))}{dt} = d\mathbf{r}_{\mathbf{q}}(\lambda, \mu).$$

Conversely, if  $\mathbf{w}$  is a tangent vector to  $S$  at  $\mathbf{p}$ , take a curve  $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbf{r}(U) \subset S$  with  $\alpha(0) = \mathbf{p}$  and  $\alpha'(0) = \mathbf{w}$ . Then take

$$\beta(t) = \mathbf{r}^{-1} \circ \alpha(t), \quad \text{so} \quad \alpha(t) = \mathbf{r} \circ \beta(t).$$

We have

$$\mathbf{w} = \alpha'(0) = d\mathbf{r}_{\mathbf{q}}\beta'(0).$$

So  $\mathbf{w}$  is in  $d\mathbf{r}_{\mathbf{q}}(\mathbb{R}^2)$ .

**Definition.** The *tangent plane* to  $S$  at  $\mathbf{p}$  is the plane which passes through  $\mathbf{p}$  and contains the tangent vectors. It is denoted by  $T_{\mathbf{p}}S$ .

Definition. If  $\mathbf{r} : U \rightarrow S$ ,

$$\mathbf{r} : (u, v) \rightarrow (x(u, v), y(u, v), z(u, v))$$

is a parameterization and  $\mathbf{r}(\mathbf{q}) = \mathbf{p}$ , then

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

is the basis of  $T_{\mathbf{p}}S$  associated to  $\mathbf{r}$ .

To write a general vector  $\mathbf{w} \in T_{\mathbf{p}}S$  in the form  $\lambda \partial \mathbf{r} / \partial u + \mu \partial \mathbf{r} / \partial v$ , we take a curve  $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbf{r}(U) \subset S$  with  $\alpha(0) = \mathbf{p}$ ,  $\alpha'(0) = \mathbf{w}$ , and set

$$\beta = \mathbf{r}^{-1} \circ \alpha.$$

Then  $\beta(0) = \mathbf{q}$  and if  $\beta'(0) = (\lambda, \mu)$ ,

$$\alpha'(0) = (\mathbf{r} \circ \beta)'(0) = \left. \frac{\partial \mathbf{r}}{\partial u} \right|_{\mathbf{q}} \left. \frac{du}{dt} \right|_0 + \left. \frac{\partial \mathbf{r}}{\partial v} \right|_{\mathbf{q}} \left. \frac{dv}{dt} \right|_0 = \lambda \left. \frac{\partial \mathbf{r}}{\partial u} \right|_{\mathbf{q}} + \mu \left. \frac{\partial \mathbf{r}}{\partial v} \right|_{\mathbf{q}}.$$

In practice this is often not the best method.

**Example.** Calculate the equation of the tangent plane to the cylinder  $x^2 + y^2 = 1$  by using the parameterization

$$\mathbf{r} : (\theta, z) \rightarrow (\cos \theta, \sin \theta, z).$$

Show that  $(3, -3, 5)$  is a tangent vector at  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$  and write it in the associated basis.

*Solution.*

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-\sin \theta, \cos \theta, 0), \quad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

The normal to the tangent plane is

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos \theta, \sin \theta, 0).$$

At  $(\theta_0, z_0)$  the tangent plane is

$$\cos \theta_0(x - \cos \theta_0) + \sin \theta_0(y - \sin \theta_0) = 0.$$

This can also be given in terms of  $(x_0, y_0, z_0)$  as

$$x_0(x - x_0) + y_0(y - y_0) = 0, \quad \text{or} \quad x_0x + y_0y = 1.$$

At  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$  it becomes the plane  $x + y = \sqrt{2}$  with normal  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ . This is indeed perpendicular to  $(3, -3, 5)$  which is therefore a tangent vector.

$$\frac{\partial \mathbf{r}}{\partial \theta} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

We solve

$$3 = -\frac{1}{\sqrt{2}}\lambda, \quad -3 = \frac{1}{\sqrt{2}}\lambda, \quad 5 = \mu,$$

to obtain

$$(3, -3, 5) = -3\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + 5(0, 0, 1).$$

Hence the vector  $(3, -3, 5)$  has coordinates  $(-3\sqrt{2}, 5)$  in the basis associated to  $\mathbf{r}$ .

**Lemma.** If  $f : U \rightarrow \mathbb{R}$  is smooth on the open set  $U \subset \mathbb{R}^3$  and  $f(\mathbf{p}) = a$  where  $a$  is a regular value of  $f$ , Then the vector  $df_{\mathbf{p}}$  is normal to the regular surface  $f = a$  at  $\mathbf{p}$ .

What different sorts of **geometry** are there?

**Lorentzian** - associated to general relativity

**Riemannian** - what we are doing here

**Symplectic** - associated to mechanics

**Complex**, including **algebraic**, **Kähler**