FINAL EXAM QUESTIONS FROM PREVIOUS YEARS. (SOME FROM OPEN BOOK EXAMS.)

Curves.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function. Consider the plane curve $\gamma : \mathbf{R} \to \mathbf{R}^2, \gamma(t) = (t, f(t))$. Prove a formula for the curvature of γ at each point $\gamma(t_0)$. Use your result to determine the curvature of the parabolas $y = kx^2$ at the origin.

2. Suppose α is a curve in $\mathbf{R}^3$ parametrized by arclength with curvature $k \neq 0$ and torsion τ. Show that the curvature of $\beta = \alpha'$ is given by $\sqrt{1 + \frac{\tau^2}{k^2}}$.

Tangent space and derivatives of maps.

1. Consider the cylinder $S = \{(x, y, z) \in \mathbf{R}^3 : x^2 + z^2 = 1\}$. Introduce coordinates (θ, z) by $x = \cos(\theta), y = \sin(\theta), z = z$. For each integer n consider the map $\varphi_n : S \to S$ which in the above coordinates takes the form $(\theta, z) \mapsto (n\theta, z)$. Determine the differential $d(\varphi_n)_{(1,0,0)}$ of φ_n at $(1, 0, 0)$ and the image of the tangent vector $(1, 0, 1)$ at this point under $d(\varphi_n)_{(1,0,0)}$.

General second fundamental form questions.

1. Let A, B and C be real numbers. Determine the second fundamental form at $(0, 0, 0)$ of the regular surface $z = Ax^2 + Bxy + Cy^2$ in $\mathbf{R}^3$, and its Gauss and mean curvature there. Also express the normal curvature along a direction $(x, y, 0) \in T_0S$ as a function of A, B, C and the angle θ between $(1, 0, 0)$ and $(x, y, 0)$.

2. Let S be a regular surface. Prove that if a point $p \in S$ is elliptic, then there is a neighbourhood U of p in S such that all the points of U are elliptic. Is the same true for hyperbolic, parabolic, or planar points? Give examples!

3. At a certain point P on the surface S, the first and second fundamental forms are: $E = 2$, $F = 1$, $G = 1$, and $e = 4$, $f = 1$, $g = 1$. Find the angle θ in the tangent plane $T_{\mathbf{p}}S$ between $\mathbf{x}_u$ and the principal direction $\mathbf{e}_1$.

Gauss curvature questions.

1. Assume that $\mathbf{x}(u, v)$ parameterizes a regular surface with unit normal $\mathbf{N}$, Gaussian curvature K and mean curvature H. Assume furthermore that all the coordinate curves

are lines of curvature. Consider the *parallel surface* $\mathbf{y}(u,v) = \mathbf{x}(u,v) + c\mathbf{N}(u,v)$ and assume that it is regular.
(a). Show that $\mathbf{N}$ is a unit normal to y;
(b). Show that $\mathbf{y}_u \times \mathbf{y}_v = (1 - 2Hc + Kc^2)\mathbf{x}_u \times \mathbf{x}_v$;
(c). Use the interpretation of the Gaussian curvature in terms of areas and the Gauss map to show that the Gaussian curvature of $\mathbf{y}$ is $K/(1 - 2Hc + Kc^2)$.

2. Give examples of surfaces with constant Gauss curvature $K = -1$, $K = 0$, and $K = 1$. Determine all surfaces of revolution with constant Gauss curvature 0.

3. Let $\alpha(s)$ be a unit-speed curve with torsion $\tau(s) \neq 0$ and binormal vector $\mathbf{b}(s)$. Let M be the ruled surface $\mathbf{x}(s,u) = \alpha(s) + u\mathbf{b}(s)$. Show that the Gaussian curvature of M at the point (s,u) is

$$K = \frac{\tau^2(s)}{(1 + u^2\tau(s))^2}.$$

Isometric surfaces.

1. Consider the following surfaces $S_i \subset \mathbf{R}^3$:

$$S_1 = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 = 1\},$$
$$S_2 = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\},$$
$$S_3 = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 - z^2 = 1\},$$
$$S_4 = \{(x,y,z) \in \mathbf{R}^3 : x = 1\},$$
$$S_5 = \{(x,y,z) \in \mathbf{R}^3 : (z-1)^2 - y^2 - x^2 = 0\}.$$

Obviously, $p = (1,0,0)$ is a point of all of these surfaces. Decide which ones are locally isometric at p (that is, for which S_i, S_j does there exist an isometry $\phi : S_i \to S_j$ defined in a neighbourhood U of p in S with $\phi(p) = p$?). Give reasons in each case. If you find two of these surfaces are isometric, give an explicit isometry.

2. Let $\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ be the standard parameterization for a surface of revolution. We know that the rotation $(u,v) \to (u+\theta, v)$ is an isometry for any angle θ. Suppose that the map $(u,v) \to (u, v+d)$, "move a distance d along the generating curve", is also an isometry for all values of d. Show that $\mathbf{x}$ must be a circular cylinder.