

**SOLUTIONS TO FINAL EXAM
QUESTIONS FROM PREVIOUS YEARS.**

Curves.

1. Let s be the arclength parameter.

$$\begin{aligned}\gamma'(t) &= (1, f'). \\ \frac{ds}{dt} &= |\gamma'(t)| = \sqrt{1 + f'^2}. \\ \frac{d\gamma}{ds} &= \frac{1}{\sqrt{1 + f'^2}}(1, f') \\ \frac{d}{dt} \frac{d\gamma}{ds} &= \frac{1}{\sqrt{1 + f'^2}}(0, f'') - \frac{f' f''}{(1 + f'^2)^{3/2}}(1, f') \\ &= \frac{1}{(1 + f'^2)^{3/2}}(-f' f'', f'') \\ \frac{d^2\gamma}{ds^2} &= \frac{f''}{(1 + f'^2)^2}(-f', 1) \\ \text{curvature} &= \left| \frac{d^2\gamma}{ds^2} \right| = \frac{f''}{(1 + f'^2)^{3/2}}\end{aligned}$$

Curvature of the parabola $y = kx^2$ is

$$\frac{2k}{(1 + 4k^2x^2)^{3/2}}.$$

2. Let t be the arclength parameter for α and let s be the arclength parameter for $\beta(t) =$

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$\alpha'(t)$. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be the Frenet frame for α .

$$\begin{aligned}\frac{d\beta}{dt} &= \frac{d^2\alpha}{dt^2} = k\mathbf{n}. \\ \frac{ds}{dt} &= \left| \frac{d\beta}{dt} \right| = k. \\ \frac{d\beta}{ds} &= \mathbf{n}. \\ \frac{d}{dt} \frac{d\beta}{ds} &= \frac{d\mathbf{n}}{dt} = -k\mathbf{t} - \tau\mathbf{b}. \\ \frac{d^2\beta}{ds^2} &= \frac{-k\mathbf{t} - \tau\mathbf{b}}{k} = -\mathbf{t} - \frac{\tau}{k}\mathbf{b}. \\ \text{curvature of } \beta &= \left| \frac{d^2\beta}{ds^2} \right| = \sqrt{1 + \frac{\tau^2}{k^2}}.\end{aligned}$$

Tangent space and derivatives of maps.

1. There are misprints in this question. Let us assume that the cylinder is $x^2 + y^2 = 1$ and that the tangent vector in $T_{(1,0,0)}$ is $(0, 1, 1)$. In the given coordinates (a coordinate map is just the inverse of a parameterization), $d\varphi_n$ has matrix

$$\begin{pmatrix} \frac{\partial n\theta}{\partial \theta} & \frac{\partial n\theta}{\partial z} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

Now $(1, 0, 0)$ corresponds to $(\theta, z) = (0, 0)$. We see that $\varphi_n(1, 0, 0) = (1, 0, 0)$. The basis for the tangent space associated to the coordinates is

$$\frac{\partial(x, y, z)}{\partial \theta} = (-\sin \theta, \cos \theta, 0), \quad \frac{\partial(x, y, z)}{\partial z} = (0, 0, 1).$$

At $(\theta, z) = (0, 0)$ this basis is

$$(0, 1, 0), \quad (0, 0, 1).$$

Then at $(1, 0, 0)$,

$$d\varphi_n(0, 1, 0) = (0, n, 0), \quad d\varphi(0, 0, 1) = (0, 0, 1).$$

Hence

$$d\varphi_n(0, 1, 1) = (0, n, 1).$$

General second fundamental form questions.

1. Following Do Carmo, for the graph $z = f(x, y)$, the matrix of the second fundamental form with respect to the basis

$$(1, 0, f_x), \quad (0, 1, f_y),$$

is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

In our case, the basis at $(0, 0, 0)$ is

$$(1, 0, 0), \quad (0, 1, 0)$$

and the matrix of the second fundamental form is

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix}.$$

The matrix of the derivative of the Gauss map is

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}^{-1} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix},$$

which gives

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{1 + f_x^2 + f_y^2}, \quad H = \frac{(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy}}{2(1 + f_x^2 + f_y^2)}.$$

In our case,

$$K = 4AC - B^2, \quad H = 2A + 2C.$$

The unit vector in direction $(x, y, 0)$ is $(\cos \theta, \sin \theta, 0) = \cos \theta(1, 0, 0) + \sin \theta(0, 1, 0)$. The normal curvature in this direction is

$$(\cos \theta, \sin \theta) \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = 2A \cos^2 \theta + 2B \cos \theta \sin \theta + 2C \sin^2 \theta.$$

2.

$$\text{elliptic points} = \{\mathbf{q} : K(\mathbf{q}) \in (0, \infty)\}.$$

Since K is a continuous (indeed a smooth) function on S and $(0, \infty)$ is open, the set of elliptic points of S is open, and is hence an open neighborhood of each of its points. The same is true for the hyperbolic points, but not for parabolic or planar points. For example, the graph

$$z = x^4 + y^2$$

is parabolic at $(0, 0, 0)$ and elliptic elsewhere, while the graph

$$z = x^4 + y^4$$

is planar at $(0, 0, 0)$ and elliptic elsewhere.

Gauss curvature questions.

1. (a).

$$\mathbf{y}_u = \mathbf{x}_u + c\mathbf{N}_u \quad \Rightarrow \quad \langle \mathbf{y}_u, \mathbf{N} \rangle = \langle \mathbf{x}_u, \mathbf{N} \rangle + c\langle \mathbf{N}_u, \mathbf{N} \rangle = \langle \mathbf{N}_u, \mathbf{N} \rangle = \frac{1}{2} \frac{\partial}{\partial u} \langle \mathbf{N}, \mathbf{N} \rangle = 0.$$

Similarly, \mathbf{N} is perpendicular to \mathbf{y}_v , and hence is normal to the surface parameterized by \mathbf{y} .

(b). Since \mathbf{x}_u and \mathbf{x}_v are lines of curvature, we have

$$\mathbf{N}_u = \mu_1 \mathbf{x}_u, \quad \mathbf{N}_v = \mu_2 \mathbf{x}_v$$

for smooth scalar functions μ_1, μ_2 (we don't specify which is bigger.)

$$\begin{aligned} \mathbf{y}_u \times \mathbf{y}_v &= (\mathbf{x}_u + c\mathbf{N}_u) \times (\mathbf{x}_v + c\mathbf{N}_v) = \mathbf{x}_u \times \mathbf{x}_v + c\mathbf{x}_u \times \mathbf{N}_v + c\mathbf{N}_u \times \mathbf{x}_v + c^2\mathbf{N}_u \times \mathbf{N}_v \\ &= \mathbf{x}_u \times \mathbf{x}_v - c(\mu_1 + \mu_2)\mathbf{x}_u \times \mathbf{x}_v + c^2\mu_1\mu_2\mathbf{x}_u \times \mathbf{x}_v = (1 - 2cH + c^2K)\mathbf{x}_u \times \mathbf{x}_v. \end{aligned}$$

(c). We think of \mathbf{N} as being a function of (u, v) . Since \mathbf{x} and \mathbf{y} have the same normal, \mathbf{N} gives the Gauss map for either surface. Consider (u_0, v_0) and set $\mathbf{x}(u_0, v_0) = \mathbf{x}_0$ and $\mathbf{y}(u_0, v_0) = \mathbf{y}_0$. Consider a small disc R around (u_0, v_0) . Then

$$\begin{aligned} \text{curvature of } \mathbf{x} \text{ at } \mathbf{x}_0 &= \lim_{\text{diam } R \rightarrow 0} \frac{\text{area } \mathbf{N}(R)}{\text{area } \mathbf{x}(R)}, \\ \text{curvature of } \mathbf{y} \text{ at } \mathbf{y}_0 &= \lim_{\text{diam } R \rightarrow 0} \frac{\text{area } \mathbf{N}(R)}{\text{area } \mathbf{y}(R)}. \end{aligned}$$

Hence

$$\frac{\text{curvature of } \mathbf{y}}{\text{curvature of } \mathbf{x}} = \lim_{\text{diam } R \rightarrow 0} \frac{\text{area } \mathbf{x}(R)}{\text{area } \mathbf{y}(R)} = \lim_{\text{diam } R \rightarrow 0} \frac{\int_R |\mathbf{x}_u \times \mathbf{x}_v| \, dudv}{\int_R |\mathbf{y}_u \times \mathbf{y}_v| \, dudv} = \frac{1}{1 - 2cH + c^2K}.$$

2. $K = 1$: the sphere $x^2 + y^2 + z^2 = 1$. $K = 0$: the plane $z = 0$. $K = -1$: the surface of revolution obtained by revolving $(e^{-s}, 0, \int_0^s \sqrt{1 - e^{-2s}} ds)$, for $s > 0$, about the z -axis.

We derived the formula for the curvature surface obtained by revolving the curve $(\phi(s), 0, \psi(s))$ around the z -axis in the case when s is the arclength - that is $\phi'^2 + \psi'^2 = 1$. The Gauss curvature is

$$-\frac{\phi''}{\phi}.$$

In particular it is zero precisely when $\phi'' = 0$. This implies that

$$\phi(s) = as + b, \quad \psi(s) = \sqrt{1 - a^2}s + c.$$

This means that the curve is a straight line and the surface of revolution is a cone or cylinder.

3.

$$\mathbf{x}_s = \alpha'(s) + u\mathbf{b}'(s) = \mathbf{t}(s) + u\tau\mathbf{n}(s), \quad \mathbf{x}_u = \mathbf{b}(s),$$

so

$$E = 1 + u^2\tau^2$$

$$F = 0$$

$$G = 1$$

Furthermore

$$\mathbf{x}_s \times \mathbf{x}_u = (\mathbf{t} + u\tau\mathbf{n}) \times \mathbf{b} = -\mathbf{n} + u\tau\mathbf{t}, \quad \mathbf{N} = \frac{-\mathbf{n} + u\tau\mathbf{t}}{\sqrt{1 + u^2\tau^2}}.$$

$$g = \langle \mathbf{x}_{uu}, \mathbf{N} \rangle = \langle 0, \mathbf{N} \rangle = 0.$$

$$f = \langle \mathbf{x}_{uv}, \mathbf{N} \rangle = \langle \tau\mathbf{n}, \frac{-\mathbf{n} + u\tau\mathbf{t}}{\sqrt{1 + u^2\tau^2}} \rangle = \frac{-\tau}{\sqrt{1 + u^2\tau^2}}.$$

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{\tau^2}{(1 + u^2\tau^2)^2}.$$

Isometric surfaces.

1. Done in class.

2. Consider the parameterizations $\mathbf{x}(u, v)$ and

$$\bar{\mathbf{x}}(u, v) = \mathbf{r}(u, v + d) = (f(v + d) \cos u, f(v + d) \sin u, g(v + d)).$$

Then

$$\bar{\mathbf{r}} \circ \mathbf{r}^{-1}$$

is the map $(u, v) \rightarrow (u, v + d)$. This is an isometry if and only if $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$.
But

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{pmatrix},$$

$$\begin{pmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{pmatrix} = \begin{pmatrix} f^2(v+d) & 0 \\ 0 & (f'(v+d))^2 + (g'(v+d))^2 \end{pmatrix}.$$

Equating E and \bar{E} we get $f(v+d) = \pm f(v)$ for all d . Assuming f is not identically zero, this gives that f is constant and the surface is a cylinder.