PRACTICE MIDTERM 1.

1. (a). The curvature is $|d\mathbf{t}/ds| = |d^2\mathbf{r}/ds^2|$. We calculate this using the chain rule.

$$\frac{d\mathbf{r}}{dt} = (-2\sin t, \cos t, 0), \qquad \frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = (4\sin^2 t + \cos^2 t)^{1/2}.$$

Hence

$$\frac{d\mathbf{r}}{ds} = \frac{dt}{ds}\frac{d\mathbf{r}}{dt} = \frac{1}{ds/dt}\frac{d\mathbf{r}}{dt} = (4\sin^2 t + \cos^2 t)^{-1/2}(-2\sin t, \cos t, 0).$$

$$\begin{aligned}\frac{d^2\mathbf{r}}{ds^2} &= \frac{dt}{ds}\frac{d}{dt}\frac{d\mathbf{r}}{ds} = (4\sin^2 t + \cos^2 t)^{-1/2}\frac{d}{dt}\Big((4\sin^2 t + \cos^2 t)^{-1/2}(-2\sin t, \cos t, 0)\Big).\\ &= (4\sin^2 t + \cos^2 t)^{-1/2}\\ &\quad\times\Big(-\frac{1}{2}(4\sin^2 t + \cos^2 t)^{-3/2}(8\sin t\cos t - 2\cos t\sin t)(-2\sin t, \cos t, 0)\\ &\qquad\qquad + (4\sin^2 t + \cos^2 t)^{-1/2}(-2\cos t, -\sin t, 0)\Big)\end{aligned}$$

at $t = 0$, this equals

$$(-2, 0, 0)$$

and so

$$\kappa_{t=0} = |(-2, 0, 0)| = 2.$$

(b). The torsion is zero because the ellipse lies in a plane.

2.

(a). The set is given by $f = 0$ where $f = xyz - e^x$. Now

$$df = (yz - e^x, xz, xy).$$

If this is $(0, 0, 0)$ then $xz = xy = 0$ so either $x = 0$ or $x \neq 0$ in which case $y = z = 0$. If $x = 0$ then $f = xyz - e^x = -1$. If $y = z = 0$ then $yz - e^x = -e^x \neq 0$ so (x, y, z) is not critical for f. Hence 0 is a regular value of f and the set $f = 0$ is a regular surface.

(b). We recognize that this is a double cone with cone point $(0,0,0)$. It is not a regular surface because there is no neighborhood of the origin in which it is a graph above one of the coordinate planes. Indeed, solving for z one gets

$$z = \pm\sqrt{x^2+y^2}.$$

Hence z is double valued close to $(0,0,0)$. On the other hand

$$y = \pm\sqrt{z^2-x^2}$$

For example there is no value of y for $z=\varepsilon$, $x=2\varepsilon$ for $\varepsilon>0$ small, while there are two values for $z=2\varepsilon$, $x=\varepsilon$, so the surface is not locally a graph over the xz plane. Similarly for the yz plane.

3. Set $f = x^2+y^2-z^2$. Then

$$df = (2x, 2y, -2z)$$

which equals $(0,0,0)$ only at the origin. At the origin $f=0$ so 1 is a regular value of f and $f=1$ is a regular surface.

To show that

$$(t,\theta) \;\rightarrow\; (\sqrt{1+t^2}\cos\theta, \sqrt{1+t^2}\sin\theta, t) \qquad\qquad 0<\theta<2\pi, \quad -\infty<t<\infty$$

is a parameterization we must check

1. It is smooth

$2'$. It is one-to-one.

3. Its derivative is one-to-one.

(1) is clear - all the components are smooth.

For (2), Suppose

$$(\sqrt{1+t_1^2}\cos\theta_1, \sqrt{1+t_1^2}\sin\theta_1, t_1) = (\sqrt{1+t_2^2}\cos\theta_2, \sqrt{1+t_2^2}\sin\theta_2, t_2).$$

Then $t_1 = t_2$ and $\sin\theta_1 = \sin\theta_1$, $\cos\theta_1 = \cos\theta_2$ so $\theta_1=\theta_2$.

For (3), the matrix of the derivative of the map is

$$\begin{pmatrix} \frac{t}{\sqrt{1+t^2}}\cos\theta & -\sqrt{1+t^2}\sin\theta \\ \frac{t}{\sqrt{1+t^2}}\sin\theta & \sqrt{1+t^2}\cos\theta \\ 1 & 0 \end{pmatrix}.$$

To find out whether it is one-to-one we calculate the cross product of the columns and we get

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{t}{\sqrt{1+t^2}}\cos\theta & \frac{t}{\sqrt{1+t^2}}\sin\theta & 1 \\ -\sqrt{1+t^2}\sin\theta & \sqrt{1+t^2}\cos\theta & 0 \end{vmatrix} = (-\sqrt{1+t^2}\cos\theta, -\sqrt{1+t^2}\sin\theta, t).$$

If this equals $(0,0,0)$ then $\sin\theta = \cos\theta = 0$ which is a contradiction, so it does not equal $(0,0,0)$ and the derivative is one-to-one.

4. The map

$$\mathbf{r} : (x,y,z) \;\to\; \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

is smooth from the cylinder $x^2+y^2=1$ to the sphere $x^2+y^2+z^2=1$. Indeed, $\mathbf{r}$ defines a smooth function on $\mathbb{R}^3 \setminus \{(0,0,0)\}$. This is an open set containing the cylinder. Moreover, it maps the cylinder onto the sphere since

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2 \;=\; 1.$$

Hence it is a smooth map from the cylinder to the sphere.