

## PRACTICE MIDTERM 2 SOLUTIONS.

1. The equation of the surface can be written  $xy - z = 0$ , so the normal is  $\nabla(xy - z) = (y, x, -1)$ . The points  $(x, y, z)$  in the tangent plane at  $(x_0, y_0, z_0)$  satisfy the equation

$$y_0(x - x_0) + x_0(y - y_0) - (z - z_0) = 0, \quad \text{or} \quad y_0x + x_0y - z = 2x_0y_0 - z_0 = z_0.$$

So  $(0, 0, 0)$  is in the tangent plane if and only if  $z_0 = 0$ . The points  $\mathbf{p}$  on the hyperbolic paraboloid satisfying this are the points in the coordinate axes  $x = z = 0$  and  $y = z = 0$ .

2. The normal to the cylinder  $C$  is  $\nabla(y^2 + z^2) = (0, 2y, 2z)$ . At  $(0, 1, 0)$  this is  $(0, 2, 0)$  and since  $(-1, 0, 1) \cdot (0, 2, 0) = 0$ , we have that  $(-1, 0, 1)$  is tangent to the cylinder at  $(0, 1, 0)$ . The map  $F$  defined above is in fact smooth from  $\mathbb{R}^3 \setminus (0, 0, 0)$  to  $\mathbb{R}^3$  and at  $(0, 1, 0)$  the derivative has matrix

$$\begin{pmatrix} \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} & -\frac{xy}{(x^2+y^2+z^2)^{3/2}} & -\frac{xz}{(x^2+y^2+z^2)^{3/2}} \\ -\frac{xy}{(x^2+y^2+z^2)^{3/2}} & \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{y^2}{(x^2+y^2+z^2)^{3/2}} & -\frac{yz}{(x^2+y^2+z^2)^{3/2}} \\ -\frac{xz}{(x^2+y^2+z^2)^{3/2}} & -\frac{yz}{(x^2+y^2+z^2)^{3/2}} & \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$dF_{(0,1,0)} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

However, for  $\mathbf{p} \in C$ ,

$$d(F|_C)_{\mathbf{p}} = dF|_{T_{\mathbf{p}}C}.$$

So  $(-1, 0, 1)$  is in fact the answer.

3. (a).

$$\frac{\partial \mathbf{r}}{\partial s} = (\rho'(s) \cos \theta, \rho'(s) \sin \theta, z'(s)), \quad \frac{\partial \mathbf{r}}{\partial \theta} = (-\rho(s) \sin \theta, \rho(s) \cos \theta, 0).$$

$$\begin{aligned}
E &= \left\langle \frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{r}}{\partial s} \right\rangle = (\rho'(s))^2 \cos^2 \theta + (\rho'(s))^2 \sin^2 \theta + (z'(s))^2 = 1, \\
F &= \left\langle \frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{r}}{\partial \theta} \right\rangle = -\rho'(s)\rho(s) \cos \theta \sin \theta + \rho'(s)\rho(s) \sin \theta \cos \theta = 0, \\
G &= \left\langle \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \theta} \right\rangle = \rho^2(s) \sin^2 \theta + \rho^2(s) \cos^2 \theta = \rho^2(s),
\end{aligned}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2(s) \end{pmatrix}.$$

(b). We need precisely that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so  $\rho \equiv 1$ . To justify this solution, suppose

$$\alpha(t) = (s(t), \theta(t)) \in \mathbb{R} \times (-\pi, \pi), \quad 0 \leq t \leq T,$$

is a smooth curve. The length of  $\mathbf{r}$  is

$$\int_0^T \sqrt{(s'(t))^2 + (\theta'(t))^2} dt.$$

The length of  $\mathbf{r} \circ \alpha$  is

$$\int_0^T \sqrt{(s'(t))^2 + \rho^2(s(t))(\theta'(t))^2} dt.$$

If  $\rho \equiv 1$  then these are equal. Conversely if the lengths are always equal and  $\rho(s_0) \neq 1$ , then choosing  $s(t) \equiv s_0$  gives

$$\int_0^T |\theta'(t)| dt = \rho(s_0) \int_0^T |\theta'(t)| dt$$

for all smooth  $\theta$ , which is a contradiction. So  $\rho \equiv 1$ .

4. Consider the parameterization of the cone  $S$  given by

$$\mathbf{r}(\theta, z) = (z \cos \theta, z \sin \theta, z), \quad z > 0, \quad -\pi < \theta < \pi.$$

(a). Show that the vectors

$$\mathbf{e}_1 = \frac{1}{z} \frac{\partial \mathbf{r}}{\partial \theta}, \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \frac{\partial \mathbf{r}}{\partial z}$$

form an orthonormal base for the tangent space of  $S$  at  $\mathbf{r}(\theta, z)$ .

(b). Calculate a smoothly varying unit normal  $\mathbf{N}$  at  $\mathbf{r}(\theta, z)$ .

(c). Calculate  $d\mathbf{N}(\mathbf{e}_1)$  and  $d\mathbf{N}(\mathbf{e}_2)$ . Hence calculate the the eigenvalues of  $-d\mathbf{N}$  (these are the principal curvatures).

(a).

$$\mathbf{e}_1 = \frac{1}{z} \frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{z} (-z \sin \theta, z \cos \theta, 0) = (-\sin \theta, \cos \theta, 0) \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \frac{\partial \mathbf{r}}{\partial z} = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, 1)$$

$$|(-\sin \theta, \cos \theta, 0)| = 1, \quad \left| \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, 1) \right| = 1, \quad (-\sin \theta, \cos \theta, 0) \cdot \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, 1) = 0.$$

(b).

$$\mathbf{N} = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ \frac{1}{\sqrt{2}} \cos \theta & \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} (\cos \theta, \sin \theta, -1).$$

(c).

$$d\mathbf{N}(\mathbf{e}_1) = \frac{1}{z} d\mathbf{N} \frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{z} \frac{\partial \mathbf{N} \circ \mathbf{r}}{\partial \theta} = \frac{1}{\sqrt{2}z} (-\sin \theta, \cos \theta, 0) = \frac{1}{\sqrt{2}z} \mathbf{e}_1,$$

$$d\mathbf{N}(\mathbf{e}_2) = \frac{1}{\sqrt{2}} d\mathbf{N} \frac{\partial \mathbf{r}}{\partial z} = \frac{1}{\sqrt{2}} \frac{\partial \mathbf{N} \circ \mathbf{r}}{\partial z} = (0, 0, 0) = 0\mathbf{e}_2.$$

Hence the eigenvalues of  $-d\mathbf{N}$  are  $-1/(\sqrt{2}z)$  and 0. (If we had chosen the other unit normal, then this would change the sign of the first eigenvalue.)