Solutions to MATH 20E FINAL EXAM Fall 98, Lindblad, Problems 1-5 only.

1. (a) $\nabla f = 6xye^{x^2y-2}\mathbf{i} + 3x^2e^{x^2y-2}\mathbf{j}$ so it increases most in the direction $12\mathbf{i} + 3\mathbf{j}$.
b) $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ so $z - 3 = 12(x-1) + 3(y-2)$.

2. a) $\nabla \times \mathbf{F} = 0$.

$$b) \quad \int_C \mathbf{F}\cdot d\mathbf{R} = \int_C \frac{xdx + ydy}{x^2+y^2} = \int_0^{2\pi} \frac{r\cos t\, r(-\sin t) + r\sin t\, r\cos t}{r^2} dt = 0.$$

c) $\mathbf{F} = \nabla\phi$, where $\phi = \ln\sqrt{x^2+y^2}$.
d) The flow lines are lines from the origin: $\dfrac{dx}{x/(x^2+y^2)} = \dfrac{dy}{y/(x^2+y^2)}$ so $\dfrac{dx}{x} = \dfrac{dy}{y}$ and hence $\ln|x| + C = \ln|y|$ which is equivalent to $|y| = e^C|x|$.

3. a) $\nabla \times \mathbf{F} = 0$.

$$b) \quad \int_C \mathbf{F}\cdot d\mathbf{R} = \int_C \frac{ydx - xdy}{x^2+y^2} = \int_0^{2\pi} \frac{r\sin t\, r(-\sin t) - r\cos t\, r\cos t}{r^2} dt = -\int_0^{2\pi} dt = -2\pi$$

c) $\mathbf{F}$ is not conservative since the integral around the closed curve in (b) is $\neq 0$.
d) The flow lines are circles centered at the origin: $\dfrac{dx}{y/(x^2+y^2)} = \dfrac{dy}{-x/(x^2+y^2)}$ so $-xdx = ydy$ and hence $y^2 = -x^2 + C$ which is equivalent to $x^2 + y^2 = C$.

4. a) $F(0,2) = (2,0)$, $F(0,4) = (0,4)$, $F(\pi/2,2) = (0,2)$ and $F(\pi/2,4) = (0,4)$.
b) $\dfrac{\partial(x,y)}{\partial(u,v)} = 2v$.

$$c) \quad \text{Area} = \iint_S dxdy = \iint_R \frac{\partial(x,y)}{\partial(u,v)} dudv = \int_0^{\pi/2}\int_2^4 2v\,dudv = \ldots = \frac{\pi}{2}\left(4^2 - 2^2\right)$$

5. a) $0 \le z \le 1$ implies that $3 \le x^2+y^2 \le 4$ so we can write S in the form of a graph $z - 1 = g(x,y) = -\sqrt{4-x^2-y^2}$ where $(x,y) \in D = \{(x,y);\ 3 \le x^2+y^2 \le 4\}$.
$dS = dxdy/|\mathbf{n}\cdot\mathbf{k}| = \sqrt{1 + g_x^2 + g_y^2 dxdy} = \ldots = 2dxdy/\sqrt{4-x^2-y^2}$ so

$$\text{Area}(S) = \iint_S dS = \iint_D \frac{dxdy}{\sqrt{4-x^2-y^2}} = \int_{\sqrt{3}}^2\int_0^{2\pi} \frac{rdrd\theta}{\sqrt{4-r^2}} = -2\pi\sqrt{4-r^2}\Big|_{\sqrt{3}}^2 = 2\pi$$

$$\iiint_B x^2\,dxdydz = \int_0^2\int_0^{2\pi}\int_0^{\pi} r^2\sin^2\phi\cos^2\theta\, r\sin\phi\, d\phi d\theta dr = \int_0^2\int_0^{2\pi} -\frac{\cos^3\phi}{3}\Big|_0^{\pi}\cos^2\theta\, r^3 d\theta dr$$
$$= \frac{2}{3}\frac{r^4}{4}\Big|_0^2\int_0^{2\pi}\cos^2\theta d\theta = \frac{2}{3}4\pi$$

NOTE: THERE ARE NO SOLUTIONS TO PROBLEMS 6-9!