Math 20E. Final. Spring 2000
Selected Solutions.

6.(a). If C is positively oriented and C^1 and P and Q are C^1 scalar fields on D then

$$\int_C P dx + Q dy \;=\; \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx dy.$$

By taking $P = 0$ and $Q = x$ we get

$$\text{area } D \;=\; \int_C x\, dy.$$

(Or take $P = -y$ and $Q = x$ to get

$$\text{area } D \;=\; -\int_C y\, dx,$$

or average these two expressions.)

(b). $(x, y) = (a\cos t, b\sin t)$ with $0 \le t \le 2\pi$.

(c). The area of the ellipse is

$$\int_C x\, dy \;=\; \int_0^{2\pi} a\cos t \frac{db\sin t}{dt}\, dt \;=\; ab\int_0^{2\pi} \cos^2 t\, dt \;=\; \pi ab.$$

7. The divergence theorem tells us that the answers to (a) and (b) are equal, so we just have to compute one of the parts. For (a) we have $\nabla \cdot \mathbf{F} = 2$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \;=\; \iiint_V \nabla \cdot \mathbf{F}\, dV \;=\; 2 \text{ volume } E.$$

Now writing D for the unit disc $x^2+y^2 \le 1$, and B for the unit ball $x^2+y^2+z^2 \le 1$, we have

$$(x, y, z) \in E \qquad \Leftrightarrow \qquad (x, y) \in D \quad \text{and} \quad -2\sqrt{1-x^2-y^2} \le z \le 2\sqrt{1-x^2-y^2}.$$

Hence the volume of E is

$$\iint_D 4\sqrt{1-x^2-y^2}\, dx dy \;=\; 2 \text{ volume of } B \;=\; \frac{8\pi}{3}.$$

Hence the answers are (a). $16\pi/3$, (b). $16\pi/3$, (c). $8\pi/3$.

We can check our answer by computing the surface integral directly. Taking the gradient, the normal to S is

$$(2x, 2y, z/2),$$

so the (outward) unit normal is

$$\mathbf{n} \;=\; \frac{(x, y, z/4)}{\sqrt{x^2 + y^2 + z^2/16}}.$$

In fact, since $\mathbf{F}$ is parallel to $\mathbf{k}$, we did not need to calculate this explicitly as we shall now see. Indeed,

$$\mathbf{F} \cdot \mathbf{n} \;=\; 2z\mathbf{n} \cdot \mathbf{k}.$$

Thinking of S as a two graphs over the disc D, we have

$$dS \;=\; \frac{1}{|\mathbf{n} \cdot \mathbf{k}|} dxdy.$$

Writing S_+ for the points of S with $z \geq 0$, and S_- for the points of S with $z \leq 0$, we have $\mathbf{n} \cdot \mathbf{k} \geq 0$ on S_+ and $\mathbf{n} \cdot \mathbf{k} \leq 0$ on S_-. Hence considering S_+ as a graph over D we have

$$\mathbf{F} \cdot \mathbf{n}\, dS \;=\; 2z\, dxdy,$$

and considering S_- as a graph over D we have

$$\mathbf{F} \cdot \mathbf{n}\, dS \;=\; -2z\, dxdy.$$

Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_+} \mathbf{F} \cdot \mathbf{n}\, dS \;+\; \iint_{S_-} \mathbf{F} \cdot \mathbf{n}\, dS \;=\; 4 \iint_D \sqrt{1 - x^2 - y^2}\, dxdy.$$

We have gotten to the same expression as before. If you did not recognize that this is related to the volume of the ball, you can compute it in polars:

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} 4\sqrt{1 - r^2}\; r\, d\theta dr \;=\; \frac{16\pi}{3}(-(1 - r^2)^{3/2}|_{r=0}^{1} \;=\; \frac{16\pi}{3}.$$