Math 20E Vector Calculus

Check Website for Course Information.
HW #1 Due this Thursday in Section. 1/12 1.1:16, 1.2:8,14,23, 1.3:6,7,12,16a , 1.5:7,18, review ch 1:29.

Lecture 1: Review of 20C.

Definition. $\mathbb{R}$ denotes the real numbers, sometimes called scalars.
$\mathbb{R}^2$ denotes the set of ordered pairs (x, y) with x, y real numbers.
$\mathbb{R}^3$ denotes the set of ordered triples (x, y, z) with x, y, z real numbers.

Geometric Interpretation. After we set up a Cartesian coordinates system, an ordered triple e.g. $(1, 2, -3)$ can represent a point in space, or a vector. The difference is that points in space are fixed, while vectors are arrows which can be moved around, added together and multiplied by scalars. When $(1, 2, -3)$ represents a point in space, the numbers $1, 2, -3$ are called the *coordinates* of the point. When it represents a vector, the numbers $1, 2, -3$ are called the components of the vector. The *position vector* of the point $(1, 2, -3)$ is just the vector $(1, 2, -3)$. It is the vector joining the origin to the point.

Definition. Addition and scalar multiplication of vectors are defined as follows. $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ and $\lambda(a_1, a_2, a_3) = (\lambda a_1 + \lambda a_2 + \lambda a_3)$.

Geometric Interpretation
Example 1. Determine the vectors **C**, **D**, **E** in terms of the vectors **A**, **B**. (see the figure from the lectures).

Principle 1. The parallelogram law for addition of vectors. (see the figure from the lectures).

Principle 2. $\lambda\mathbf{A}$ is the vector parallel to **A** whose length is $|\lambda|$ times the length of **A** and which is in the same direction as **A** if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.

Solution. $\mathbf{C} = \mathbf{A} + \mathbf{B}, \qquad \mathbf{D} = \mathbf{B} - \mathbf{A}, \qquad \mathbf{E} = \frac{1}{2}(\mathbf{A} + \mathbf{B})$.

Example 2. Compute the vector represented by $\overrightarrow{AB}$, where $A = (5, 1, -2)$ and $B = (7, 8, 3)$.

Definition.

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \mathbf{k} = (0, 0, 1).$$

Then

$$(a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Example 3. Calculate the parametric and symmetric equations of the line L through the point $(2, 3, -1)$ in the direction $(1, 0, -2)$.

Solution. Now $\mathbf{R}_0 = (2, 3, -1)$ is the position vector of the point which we are given on L. If $\mathbf{R} = (x, y, z)$ is a general point on the line, then $\mathbf{R} - \mathbf{R}_0$ is parallel to the line, so there exists a scalar t such that $\mathbf{R} - \mathbf{R}_0 = t(1, 0, -2)$. Hence the equation of the line is

$$(x, y, z) \;=\; (2, 3, -1) \;+\; t(1, 0, -2).$$

Writing out the components we get

$$\begin{aligned} x &= 2 \;+\; t, \\ y &= 3, \\ z &= -1 \;-\; 2t. \end{aligned}$$

Eliminating t, the symmetric equations are

$$x \;-\; 2 \;=\; \frac{z \;+\; 1}{-2}, \qquad y = 3.$$

Definition. The length of $\mathbf{a} = (a_1, a_2, a_3)$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
The unit vector in the direction of $\mathbf{a}$ is $\mathbf{a}/\|ba\|$.
The inner product (= scalar product = dot product) of $\mathbf{a}$ and $\mathbf{b} = (b_1, b_2, b_3)$ is

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) \;=\; a_1 b_1 \;+\; a_2 b_2 \;+\; a_3 b_3.$$

Geometric Interpretation.
Example 4. Calculate the angle between the vectors $\mathbf{a} = (-4, 5, 6)$ and $\mathbf{b} = (3, 0, 4)$.

unfortunately figures will not be included in lecture notes until later in the quarter

Principle. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then

$$\mathbf{a} \cdot \mathbf{b} \;=\; |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Solution. We have

$$\cos\theta \;=\; \frac{-4 \cdot 3 \;+\; 5 \cdot 0 \;+\; 6 \cdot 4}{\sqrt{(-4)^2 + 5^2 + 6^2}\sqrt{3^2 + 0^2 + 4^2}} \;=\; \frac{12}{\sqrt{77} \cdot 5}, \qquad\qquad \theta = \cos^{-1}\left(\frac{12}{5\sqrt{77}}\right).$$

Example 5. Find a value of λ so that the vectors $\mathbf{u} = (1, -3, 5)$ and $\mathbf{v} = (2, -6, z)$ are
(a). Parallel
(b). Perpendicular.

Principle. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if one is a scalar multiple of the other. (There is also a condition in terms of the cross product).

Solution (a). Solve

$$\lambda(1,-3,5) \;=\; (-2,6,z), \qquad \Rightarrow \qquad \lambda=-2, \quad -3\lambda=6, \qquad 5\lambda=z,$$

so $\lambda = -2$ and $z = -10$.

Principle. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular exactly when $\mathbf{u}\cdot\mathbf{v}=0$.

Solution (b).

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot -2 \;+\; -3\cdot 6 \;+\; 5\lambda \;=\; -20+5z.$$

Hence $\mathbf{u}\cdot\mathbf{v}=0$ if and only if $z=4$.

Definition. The projection of $\mathbf{a}$ on $\mathbf{b}$ is the vector $\mathbf{p}$ such that $\mathbf{a}=\mathbf{p}+\mathbf{q}$ where $\mathbf{p}$ is parallel to $\mathbf{b}$ and $\mathbf{q}$ is perpendicular to $\mathbf{b}$. To compute it, write $\mathbf{p}=\lambda\mathbf{b}$.

$$\mathbf{a} \;=\; \lambda\mathbf{b}+\mathbf{q}, \qquad \Rightarrow \qquad \mathbf{a}\cdot\mathbf{b} \;=\; \lambda\mathbf{b}\cdot\mathbf{b}, \qquad \Rightarrow \qquad \lambda=\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}, \qquad \Rightarrow \qquad \mathbf{p}=\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}.$$

Example 6. Let $\mathbf{a}=(2,0,3)$ and $\mathbf{b}=(-2,2,-1)$. Calculate the projection of $\mathbf{a}$ on $\mathbf{b}$.
Solution The projection is

$$\frac{(2,0,3)\cdot(-2,2,1)}{\|(-2,2,-1)\|^2}(-2,2,-1) \;=\; \frac{-1}{9}(-2,2,-1) \;=\; \left(\frac{2}{9},\frac{-2}{9},\frac{1}{9}\right).$$