Lecture 10: Section 4.4. Divergence and Curl.

Recap gradient and divergence:

$$\nabla(xye^z) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(xye^z) = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}.$$

$$\begin{aligned}\nabla\cdot(xy\mathbf{i} - 3e^z\mathbf{j} + \sin z\mathbf{k}) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(xy\mathbf{i} - 3e^z\mathbf{j} + \sin z\mathbf{k}) \\ &= \frac{\partial(xy)}{\partial x} + \frac{\partial e^z}{\partial y} + \frac{\partial \sin z}{\partial z} = y + \cos z.\end{aligned}$$

Physical interpretation of the divergence 1.

The divergence of the velocity vector field $\mathbf{F}$ *of the fluid is the rate of expansion of the fluid per unit volume.*

A fluid moves in space. $\mathbf{F}(x, y, z)$ is the velocity of the particles at (x, y, z). The particles travel along the flow lines of $\mathbf{F}$. Take a small region R around (x, y, z) with volume V. After time t it has flowed to a new region R_t with volume V_t. Then

$$\nabla\cdot\mathbf{F}(x, y, z) = \lim_{R\to(x,y,z)} \frac{1}{V}\left(\left.\frac{dV_t}{dt}\right|_{t=0}\right).$$

If $\nabla\cdot\mathbf{F}(x, y, z) > 0$ *then the fluid at* (x, y, z) *is expanding. If* $\nabla\cdot\mathbf{F}(x, y, z) < 0$ *then the fluid at* (x, y, z) *is contracting. If the fluid is "incompressible" then* $\nabla\cdot\mathbf{F} = 0$.

Physical interpretation of the divergence 2.

Let $\mathbf{F}$ be the flow of a fluid (instead of the velocity). The flow or *flow rate density* is defined to be

$$\mathbf{F} = \mu\mathbf{v},$$

where μ is the density and $\mathbf{v}$ is the velocity. It measures the rate and direction in which mass is moving at a given point. To be precise, for a surface S with unit normal $\mathbf{n}$, we define the **flux of F across** S to be the rate in *mass per unit time* at which fluid crosses over S (in the general direction of $\mathbf{n}$). If now S is a small piece of plane with unit normal $\mathbf{n}$ which contains (x, y, z) and havs area $\triangle S$, then the **flux of F across** S is approximately

$$\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})\cdot\mathbf{n}\,\triangle S.$$

Indeed, in a small time $\triangle t$, the fluid particles close to (x, y, z) will travel approximately along the vector $\mathbf{v}\triangle S$. Those that will cross $\triangle S$ are in the sloped cylinder with $\triangle S$ as its base and $\mathbf{v}\cdot\mathbf{n}\,\triangle t$ as its side. But the volume of this sloped cylinder is $\mathbf{v}\cdot\mathbf{n}\,\triangle t\triangle S$. Hence the mass of fluid inside is $\mathbf{F}\cdot\mathbf{n}\,\triangle t\triangle S$. The rate at which mass crosses S is obtained by dividing by Δt.

The divergence of the flow $\mathbf{F}(x,y,z)$ is the rate at which $\mathbf{F}$ flows out from a small region R containing (x,y,z) divided by the volume V of the region.

$$\nabla\cdot F(x,y,z) \;=\; \lim_{R\to(x,y,z)} \frac{\text{flux of }\mathbf{F}\text{ out from of the boundary of }R}{\text{Volume of }R}.$$

If $\nabla\cdot\mathbf{F}(x,y,z)>0$ then either the density is decreasing at (x,y,z) or fluid is being created there. If $\nabla\cdot\mathbf{F}(x,y,z)<0$ then either the density is increasing at (x,y,z) or fluid is being removed there.

Curl.

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

We symbolically define the **curl** of a vector field $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ by

$$\nabla\times\mathbf{F}\;==\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_1&F_2&F_3\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_2&F_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial z}\\ F_1&F_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial y}\\ F_1&F_2\end{vmatrix}\mathbf{k}$$

$$=\Big(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\Big)\mathbf{i}+\Big(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\Big)\mathbf{j}+\Big(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\Big)\mathbf{k}$$

This final formula gives the real definition, but the symbolic formula makes it easy to remember and compute.

Example. Find $\nabla\times\mathbf{F}$ when

(i). $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$.

(ii). $\mathbf{F}=-y\mathbf{i}+x\mathbf{j}$.

(iii). $\mathbf{F}=(-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$.

Solution. (i).

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&y&0\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ y&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial z}\\ x&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial y}\\ x&y\end{vmatrix}\mathbf{k}=\mathbf{0}$$

(ii).

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ -y&x&0\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial z}\\ -y&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial y}\\ -y&x\end{vmatrix}\mathbf{k}=2\mathbf{k}$$

(iii).

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]\mathbf{k} = ... = \mathbf{0}$$

Physical Interpretation of curl. Place a small paddle wheel in a fluid with flow $\mathbf{F}$. The paddle wheel will turn fastest when its axis points in the direction of $\nabla \times \mathbf{F}$. Moreover, it will turn according to the right hand rule: if the thumb of the right hand points towards $\nabla \times \mathbf{F}$ then the fingers curl in the direction that the fluid rotates.