MATH 20E VECTOR CALCULUS

Lecture 11. The gradient operation sends scalar fields to vector fields. The divergence operation sends vector fields to scalar fields. The curl operation sends vector fields to vector fields. Are these operations related?

Theorem. For every C^2 scalar field f and C^2 vector field $\mathbf{F}$,

$$\nabla \times \nabla f = \mathbf{0}, \qquad \nabla \cdot \nabla \times \mathbf{F} = 0.$$

Proof.

$$\nabla \times \nabla f \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} \;=\; (f_{yz} - f_{yz})\mathbf{i} \;+\; \ldots \;=\; 0.$$

$$\nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} \;=\; \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \;=\; \ldots \;=\; 0.$$

Example. Consider the vector field $z(\mathbf{i}+\mathbf{k})$. Is it equal to ∇f or $\nabla \times \mathbf{F}$ for any scalar field f or vector field $\mathbf{F}$?

Solution.

$$\nabla \cdot z(\mathbf{i}+\mathbf{k}) \;=\; 1,$$

so $z(\mathbf{i}+\mathbf{k})$ is not the curl of another vector field.

$$\nabla \times z(\mathbf{i}+\mathbf{k}) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & 0 & z \end{vmatrix} \;=\; \mathbf{j}.$$

Hence $z(\mathbf{i}+\mathbf{k})$ is not the gradient of a scalar field.

Area Integrals Review.

Example 1. Calculate the area integral

$$\iint_R xy\,dA,$$

where R is the triangular region in the x–y plane bounded by the lines $x = y$, $x + y = 2$, $x = 0$. Interpret the answer and the calculation geometrically.

Solution. We first describe this region in Cartesian coordinates:

$$0 \le x \le 1, \qquad x \le y \le 2 - x.$$

Then

$$\begin{aligned}\iint_R xy\,dA &= \int_{x=0}^{1}\int_{y=x}^{2-x} xy\,dydx = \int_{x=0}^{1}\frac{x}{2}\left((2-x)^2-x^2\right)\,dx\\ &= \int_{x=0}^{1} 2x-2x^2\,dx = \left(x^2-\frac{2x^3}{3}\right)\Big|_0^1 = \frac{1}{3}.\end{aligned}$$

The volume of the region W under the graph $z = xy$ and above the planar region R in the xy plane is $1/3$. We computed this by first computing the area of the slice of W obtained by fixing a value of x. We then integrated the areas of these slices to get the volume of W.

Example 2. Calculate

$$\iint_R x\,dA,$$

where R is the region $1 \le x^2 + y^2 \le 4$ with $0 < x < y$. Interpret the answer and the calculation geometrically.

Solution. We use Polar coordinates. Describing the region in polars r, θ, we have

$$1 \le r \le 2, \qquad\qquad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}.$$

Recalling that in polars $dA = rdrd\theta$, we have

$$\begin{aligned}\iint_R x\,dA &= \int_{r=1}^{2}\int_{\theta=\pi/4}^{\pi/2} r\cos\theta\, r\,d\theta dr\\ &= \int_{r=1}^{2} r^2\left(1-\frac{1}{\sqrt{2}}\right)\,dr = \frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)(2^3-1^3).\end{aligned}$$

This number gives the volume of the region W in space which lies below the graph $z = x$ and above the region R in the xy plane. We computed this by first computing the area of the surface obtained by fixing a value of r, and then integrating with respect to r. From another perspective, the area integral is expressed in terms of polar coordinates by relating the area element $dA = dxdy$ to $drd\theta$. Indeed, the area of a small region given by $r_0 \le r \le r_0 + \triangle r$ and $\theta_0 \le \theta \le \theta_0 + \triangle\theta$ is $r_0\triangle r\triangle\theta +$ higher order terms. This gives a new $r - \theta$ area integral.