Lecture 12: Changing the order of integration

A region D in the plane is x-simple if it is the region between two graphs of the form $x = \phi(y)$, more precisely it is given by

$$a \le y \le b, \qquad \phi_1(y) \le x \le \phi_2(x).$$

The region D is y-simple if it is the region between two graphs of the form $y = \phi(x)$, more precisely it is given by

$$a \le x \le b, \qquad \phi_1(x) \le y \le \phi_2(x).$$

A **simple region** is one which is both x-simple and y-simple. An **elementary region** is one that is either x-simple or y-simple (or both).

Example. Evaluate

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy.$$

We cannot carry out the inner integral because we cannot integrate e^{x^2}. However, there is a trick which works in this case. The trick is to change the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \iint_D e^{x^2}\, dA, \quad (x,y) \in D \Leftrightarrow 0 \le y \le 1,\ 3y \le x \le 3.$$

Now we describe the region D with the variables in the other order.

$$(x,y) \in D \qquad \Leftrightarrow \qquad 0 \le x \le 3,\ 0 \le y \le x/3.$$

so

$$\iint_D e^{x^2}\, dA = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 e^{x^2} y\Big|_{y=0}^{x/3} dx = \int_0^3 e^{x^2} \frac{x}{3}\, dx = \frac{e^{x^2}}{6}\Big|_{x=0}^{3} = \frac{e^9 - 1}{6}.$$

Physical interpretation. $f > 0$.

$$\int_D f\, dA$$

represents the volume of the solid region above the region D in the xy plane and below the graph $z = f(x,y)$. It also represents the mass of a sheet of material in the region D having density f. **Areas** can be calculated using double integrals:

$$\iint_D 1\, dA = \text{Area}(D).$$

This is because the integral is the volume above D and below 1 which is $\text{Area}(D) \cdot 1$.

Triple Integrals. We now want to define the integral of a function f over a rectangular box $B = \{(x, y, z); a_0 \le x \le a,\ b_0 \le y \le b,\ c_0 \le z \le c\}$. We divide B into smaller boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ by dividing the interval $[a_0, a]$ into n subintervals of length $\triangle x = (a - a_0)/n$, $[b_0, b]$ into n subintervals of length $\triangle y$ and $[c_0, c]$ into n subintervals of length $\triangle z$. Then we form the Riemann sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\triangle V \tag{1}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in B_{ijk} and $\triangle V = \triangle x\, \triangle y\, \triangle z$. We then define the integral to be the limit of the Riemann sum:

$$\iiint_B f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\triangle V \tag{2}$$

As for the case of two variables we can write it as iterated integrals (Fubini):

$$\iiint_B f(x, y, z)\, dV = \int_{c_0}^{c} \Big(\int_{b_0}^{b} \Big(\int_{a_0}^{a} f(x, y, z)\, dx \Big)\, dy \Big)\, dz \tag{3}$$

or one can integrate the different variables in any other order.
Example. Suppose B is the box with $1 \le x \le 2$, $0 \le y \le 1$, $-1 \le z \le 2$. Find

$$\iiint_B x^2 yz\, dV.$$

Solution.

$$\iiint_B x^2 yz\, dV = \int_{-1}^{2}\int_0^1\int_1^2 x^2 yz\, dx\, dy\, dz \;=\; \int_{-1}^{2}\int_0^1 \frac{7}{3} yz\, dxdy \;=\; \int_{-1}^{2} \frac{7}{6} z\, dz \;=\; \frac{7}{4}.$$

We define the integral of f over a more general bounded region E by finding a large box B containing E and integrating the function that is equal to f in E and 0 outside E over the lager box B. We now restrict our attention to *elementary* regions of space.
Definition. The region W is **elementary** if it can be written in one of three possible ways. The first way is

$$(*) \qquad (x, y, z) \in W \qquad \Leftrightarrow \qquad (x, y) \in D, \qquad\qquad u_1(x, y) < z < u_2(x, y)$$

where D is an elementary region of the plane and ϕ_1 and ϕ_2 are bounded functions. The other two ways are similar to (*) but the roles of the variables are interchanged. As for area integrals, we calculate volume integrals by the method of iterated integrals. If W is the region given in (*) then

$$\iiint_W f(x,y,z)\,dV = \iint_D \Big(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \Big) dA.$$

Physical Interpretation of Volume integrals If $f(x,y,z) > 0$ represents the density (mass per unit volume) at the point (x,y,z), then

$$\iiint_W f\,dV$$

is the total mass of W. This integral also represents the volume of a 4-dimensional region. The volume of W is

$$\text{Volume}(W) = \iiint_E 1\,dV.$$

Example 2. Calculate $\iiint_W xy\,dV$, where W is the part of the ball $x^2+y^2+z^2 \leq 1$ with $x,y,z \geq 0$.

Solution. We can describe the region by

$$(x,y,z) \in W \qquad \Leftrightarrow \qquad (x,y) \in D,\ 0 \leq z \leq \sqrt{1-x^2-y^2},$$

where D is the quarter circle in the plane given by

$$(x,y) \in D \qquad \Leftrightarrow \qquad 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}.$$

Then

$$\begin{aligned}
\iiint_W xy\,dV &= \iint_D \int_{z=0}^{\sqrt{1-x^2-y^2}} xy\,dzdydx = \iint_D xy\sqrt{1-x^2-y^2}\,dydx \\
&= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-y^2}} xy\sqrt{1-x^2-y^2}\,dydx = \int_{x=0}^{1} \frac{-x}{3}(1-x^2-y^2)^{3/2}\Big|_{y=0}^{\sqrt{1-x^2}}\,dx \\
&= \int_{x=0}^{1} \frac{x}{3}(1-x^2)^{3/2}\,dx = \frac{-1}{15}(1-x^2)^{5/2}\Big|_{x=0}^{1} = \frac{1}{15}.
\end{aligned}$$

Putting the variables in another order, we get the same answer:

$$0 \leq z \leq 1, \qquad 0 \leq y \leq \sqrt{1-z^2}, \qquad 0 \leq x \leq \sqrt{1-y^2-z^2}.$$

Then

$$\begin{aligned}
\iiint_V xy\,dV &= \int_{z=0}^{1}\int_{y=0}^{\sqrt{1-z^2}}\int_{x=0}^{\sqrt{1-y^2-z^2}} xy\,dxdydz \\
&= \int_{z=0}^{1}\int_{y=0}^{\sqrt{1-z^2}} \frac{(1-y^2-z^2)y}{2}\,dydz \\
&= \int_{z=0}^{1}\left(\frac{(1-z^2)y^2}{4}-\frac{y^4}{8}\right)\Bigg|_{y=0}^{\sqrt{1-z^2}}\,dz \\
&= \frac{1}{8}\int_{z=0}^{1}(1-z^2)^2\,dz \\
&= \frac{1}{8}\left(\frac{6}{5}-\frac{2}{3}\right) = \frac{1}{15}.
\end{aligned}$$