Lecture 13: The change of variable theorem on the line. Suppose that $x(u)$, $a \le u \le b$ is a change of variables. In order for it to be invertible we assume that $dx(u)/du > 0$, when $a \le u \le b$. Then we can change variables in the integral:

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\frac{dx}{du}.du$$

Symbolically,

$$dx = \frac{dx}{du}du.$$

Change of variables for area integrals. Let $T(u,v) = (x,y)$;

$$x = x(u,y), \qquad y = y(u,v),$$

be a C^1 **mapping** from a nice region D^* in the u-v plane **onto** a region $D = T(D^*)$ in the x-y plane. We say that a map is **one-to-one** if no two points are mapped to one point, or formulated differently if $T(u,v) = T(u',v')$ implies that $(u,v) = (u',v')$.

Change of variable theorem in the plane.

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

We are writing everything in the integral on the left in terms of the new variables (u,v). Symbolically,

$$dxdy \;=\; \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where the determinant on the right is called the *Jacobian* of the map T. It is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; |DT| \;=\; \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Example. Compute the Jacobian of the polar coordinate map given by $T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$.

Solution.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \neq 0.$$

This agrees with our previous calculation. Indeed, we already computed that

$$dxdy \;=\; rdrd\theta.$$

Example. Let $T(u,v) = (u^2 - v^2, 2uv) = (x,y)$. Let D^* be the quarter circle in the uv plane given by $u^2 + v^2 \le 1$, $u, v \ge 0$. Describe $D = T(D^*)$ and change variables to (u,v) to compute

$$\iint_D dxdy.$$

Solution. In general it is not easy to compute what mappings do to regions. Here are some strategies for understanding mappings.

- If the map T is the linear map

$$T\begin{pmatrix} u \\ v \end{pmatrix} \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

then T is easy to understand. It maps straight lines to straight lines and maps parallelograms to parallelograms. We will understand it better shortly.

- Try to find the inverse of T to see if it is one-to-one and onto.
- See what T does to the boundary curves of D^*.
- Try to relate T to a mapping you already understand.

We will see what our map T does to the boundary of D^*. It maps the line segment $v = 0$, $0 \le u \le 1$ onto the line segment $y = 0$, $0 \le x \le 1$ and the line $u = 0$, $0 \le v \le 1$ onto the line segment $y = 0$, $-1 \le x \le 0$. For the curved piece of boundary $u^2 + v^2 = 1$, we parameterize by θ to get $(u,v) = (\cos\theta, \sin\theta)$ with $0 \le \theta \le \pi/2$. This is mapped to $(x,y) = (\cos^2\theta - \sin^2\theta, 2\cos\theta\sin\theta) = (\cos 2\theta, \sin 2\theta)$. Hence this quarter circle in the uv plane is mapped to a half circle, and D is a half disc of radius 1, so we already know that the integral represents the area of D which is $\pi/2$, but let us compute this using the change of variables. We have

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \;=\; \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \;=\; 4(u^2 + v^2),$$

so

$$\iint_D dxdy \;=\; \iint_{D^*} 4(u^2 + v^2)\, dudv.$$

Introducing polar coordinates in the uv plane, this is

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} 4r^2 rdrd\theta \;=\; \int_0^1 2\pi r^3\, dr \;=\; \frac{\pi}{2}.$$

Linear maps of the plane Consider the linear map

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au+bv \\ cu+dv \end{bmatrix} = u\begin{bmatrix} a \\ c \end{bmatrix} + v\begin{bmatrix} b \\ d \end{bmatrix}.$$

This maps the points $(1,0)$ and $(0,1)$ to (a,c) and (b,d) respectively, and it maps $u(1,0)+v(0,1)$ to $u(a,c)+v(b,d)$, so it maps the integer grid in the uv plane to a "parallelogram grid". It maps parallelograms to parallelograms, and it scales up all areas by a constant factor given by the determinant

$$\begin{vmatrix} au+b & \\ c & d \end{vmatrix}.$$

Notice that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det|DT| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

also gives the area scale factor of the mapping T.

Now the general map T is approximated close to (u,v) by the derivative DT. Indeed,

$$T(u+\triangle u, v+\triangle v) \approx T(u,v) + DT(u,v)\begin{bmatrix} \triangle u \\ \triangle v \end{bmatrix}.$$

Since the map T is approximately given by the linear map DT, close to (u,v), it scales areas by approximately the same factor as this linear map, which is the determinant

$$|DT(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Symbolically, this has the form

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

We should be able to divide through by the Jacobian and get

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}.$$

This is indeed true. To see this we note that for two matrices A and B, the determinant is multiplicative, $|A||B| = |AB|$. But then

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Example. By making the change of variables $u = e^{-y}\cos x$, $v = e^{-y}\sin x$, calculate

$$\iint_D \frac{1}{\cos^2 x}\, dxdy$$

where D is a region in the xy plane which is mapped one-to-one and onto the rectangle D^* in the uv plane given by

$$\frac{1}{4} \leq u \leq \frac{1}{2}, \qquad\qquad \frac{1}{4} \leq v \leq \frac{1}{2}.$$

Solution. The region D^* is the rectangle $1/4 < u < 1/2$, $1/4 < v < 1/2$. The map $T : (u, v) \to (x, y)$ has inverse

$$T^{-1}(x, y) \;=\; (e^{-y}\cos x, e^{-y}\sin x) = (u, v),$$

We calculate

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \;=\; \begin{vmatrix} -e^{-y}\sin x & e^{-y}\cos x \\ -e^{-y}\cos x & e^{-y}\sin x \end{vmatrix} \;=\; -e^{-2y}(\cos^2 x + \sin^2 x) \;=\; e^{-2y}.$$

Hence

$$dxdy \;=\; \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \;=\; \left|\frac{1}{e^{-2y}}\right| dudv \;=\; e^{2y}dudv.$$

Then

$$\iint_D \frac{1}{\cos^2 x}\, dxdy \;=\; \int_{D^*} \frac{e^{2y}}{\cos^2 x}\, dudv \;=\; \int_{D^*} \frac{1}{u^2}\, dudv \;=\; \int_{1/4}^{1/2}\int_{1/4}^{1/2} \frac{1}{u^2}\, dudv \;=\; \frac{1}{2}.$$