

**Lecture 14. Change of variables.**

Suppose that  $T(u, v) = (x, y)$  is a one-to-one mapping  $x = x(u, v)$ ,  $y = y(u, v)$ , from a piecewise smooth simply connected domain  $D^*$  in the  $u$ - $v$  plane onto a domain  $D = T(D^*)$  in the  $x$ - $y$  plane. Then the **change of variable theorem** states that

$$\iint_D f(x, y) \, dx \, dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} := \|DT\| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.$$

**Example.** Let  $P$  be the parallelogram with vertices  $(0, 0)$ ,  $(2, 1)$ ,  $(1, 2)$  and  $(3, 3)$ . By making a linear change of variables to simplify this region, compute

$$\iint_P xy \, dA.$$

**Solution.** The sides of  $P$  are in the lines  $2y - x = 0$ ,  $2x - y = 0$ ,  $2y - x = 3$  and  $2x - y = 3$ . Set  $u = 2y - x$  and  $v = 2x - y$ . Then if  $T(u, v) = (x, y)$  and  $T^{-1}P = P^*$ , we have that  $P^*$  is the square  $0 \leq u, v \leq 3$ , using the principle

- A linear map takes straight lines to straight lines and parallelograms to parallelograms.

We can invert the map  $T$  to get

$$x = \frac{u + 2v}{3}, \quad y = \frac{v + 2u}{3}.$$

We see that  $T : P^* \rightarrow P$  is one-to-one and onto, and

$$\iint_D xy \, dA = \int_0^3 \int_0^3 \frac{(u + 2v)(v + 2u)}{9} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

But

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{vmatrix} = -\frac{1}{3}.$$

Hence

$$\iint_D xy \, dA = \int_0^3 \int_0^3 \frac{2u^2 + 2v^2 + 5uv}{9} \left| -\frac{1}{3} \right| \, du \, dv = \frac{1}{27} \left( 4 \times 9 + \frac{5 \times 81}{4} \right).$$

Sometimes it is helpful to make use of the following result:

- A linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is one-to-one and onto provided  $\det T \neq 0$ .

In this case the inverse  $T^{-1}$  is also a linear map.

**Example from last time.** By making the change of variables  $u = e^{-y} \cos x$ ,  $v = e^{-y} \sin x$ , calculate

$$\iint_D \frac{1}{\cos^2 x} dx dy$$

where  $D$  is a region in the  $xy$  plane which is mapped one-to-one and onto the rectangle  $D^*$  in the  $uv$  plane given by

$$\frac{1}{4} \leq u \leq \frac{1}{2}, \quad \frac{1}{4} \leq v \leq \frac{1}{2}.$$

**Solution.** The region  $D^*$  is the rectangle  $1/4 < u < 1/2$ ,  $1/4 < v < 1/2$ . The map  $T : (u, v) \rightarrow (x, y)$  has inverse

$$T^{-1}(x, y) = (e^{-y} \cos x, e^{-y} \sin x) = (u, v),$$

We calculate

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -e^{-y} \sin x & e^{-y} \cos x \\ -e^{-y} \cos x & e^{-y} \sin x \end{vmatrix} = -e^{-2y}(\cos^2 x + \sin^2 x) = -e^{-2y}.$$

Hence

$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \left| \frac{1}{e^{-2y}} \right| du dv = e^{2y} du dv.$$

Then

$$\iint_D \frac{1}{\cos^2 x} dx dy = \int_{D^*} \frac{e^{2y}}{\cos^2 x} du dv = \int_{D^*} \frac{1}{u^2} du dv = \int_{1/4}^{1/2} \int_{1/4}^{1/2} \frac{1}{u^2} du dv = \frac{1}{2}.$$

**Why is the change of variables formula true?** Divide up  $D^*$  to small rectangles  $D_{ij}^*$  with vertex  $(u_i, v_j)$  and spanned by vectors  $(\Delta u_i, 0)$  and  $(0, \Delta v_j)$ . For  $(u, v)$  in this rectangle we have

$$T \begin{pmatrix} u \\ v \end{pmatrix} \approx T \begin{pmatrix} u_i \\ v_j \end{pmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} u - u_i \\ v - v_j \end{bmatrix}.$$

where the matrix  $DT$  is evaluated at  $(u^i, v^i)$ . But this approximation to  $T$  maps the rectangle  $D_{ij}^*$  onto a parallelogram  $D_{ij}$  containing  $(x_i, x_j) = (x(u_i, v_j), y(u_i, v_j))$  whose area is

$$\left| \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| \Delta u_i \Delta v_j = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u_i \Delta v_j.$$

Hence

$$\begin{aligned} \iint_D f(x, y) dA &\approx \sum_{i,j=1}^n f(x_i, y_j) \text{Area}(D_{ij}) \\ &\approx \sum_{i,j=1}^n f(x(u_i, v_j), y(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| (u_i, v_j) \Delta u_i \Delta v_j \\ &\rightarrow \iint_{D^*} f(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \end{aligned}$$

**Changing variables in Triple Integrals.** If  $T(u, v, w) = (x, y, z)$  is an invertible mapping  $x = x(u, v, w)$ ,  $y = y(u, v, w)$ ,  $z = z(u, v, w)$  and  $W = T(W^*)$  then

$$\iiint_W f(x, y, z) dx dy dz = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw,$$

where the Jacobian determinant is

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Symbolically:

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

In **spherical coordinates**  $x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $z = \rho \cos \phi$ ,  $\rho \geq 0$ ,  $0 \leq \phi \leq \pi$ ,  $0 \leq \theta < 2\pi$ , we have  $dx dy dz = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} d\rho d\phi d\theta = \dots = \rho^2 \sin \phi d\rho d\phi d\theta$