Lecture 15: Path integrals and Line Integrals. Suppose $\mathbf{c}(t)$ with $a \le t \le b$ is a C^1 path in space. Its image in space is a curve, C. It is *oriented* because the path $\mathbf{c}$ gives it a direction. We can reparameterize the curve C by taking an invertible C^1 map $h : [c,d] \to [a,b]$ and forming the curve $\mathbf{c}_2(\tau) = \mathbf{c}(h(\tau))$. If $h(c) = a$ and $h(d) = b$ then this reparameterization will preserve the orientation, while if $h(c) = b$ and $h(d) = a$ it will reverse the orientation.

Suppose $f(x,y,z)$ is a function defined at each point of the curve C. Let the curve be parameterized by arclength, so it is given by $(x(s), y(s), z(s))$ with $0 \le s \le L$ where L is the length of the curve. We define the **path integral** or **arc-length integral** of f along C to be

$$\int_C f\, ds \;=\; \int_0^L f(x(s), y(s), z(s))\, ds.$$

In terms of another parameterization $\mathbf{c}(t)$ with $a \le t \le b$ we have

$$\int_C f\, ds \;=\; \int_a^b f(x(t), y(t), z(t))\, \|\mathbf{c}'(t)\|\, dt.$$

This is just the change of variables formula. The value of this path integral does not depend on the parameterization of the curve C. It does not even change if we switch the orientation.

Geometric Interpretation. If C is a curve in the xy plane and $f > 0$ is a positive function on C, then

$$\int_C f\, ds$$

represents the area of the surface S given by

$$(x,y,z) \in S \quad \Leftrightarrow \quad (x,y) \in C, \text{ and } 0 \le z \le f.$$

Example 1. Calculate the area of the surface given by $x^2 + y^2 = 4$, $0 \le z \le xy$, $x, y \ge 0$.

Solution. We parameterize the circle $\mathbf{c}(\theta)$ by $(x,y) = (2\cos\theta, 2\sin\theta)$, where $0 \le \theta \le \pi/2$. Then setting $f(x,y) = xy$, the area of the surface is

$$\begin{aligned}\int_C f\, ds \;&=\; \int_0^{\pi/2} (2\cos\theta 2\sin\theta)\; |(-2\sin\theta, 2\cos\theta)|\; d\theta \;=\; 4\int_0^{\pi/2} \sin 2\theta\, d\theta \\ &=\; -2\cos 2\theta|_0^{\pi/2} \;=\; 4.\end{aligned}$$

Why does the path integral compute surface area? It is approximated by the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i) \triangle s_i,$$

where we have split the curve C into n small pieces C_i with length $\triangle s_i$, and (x_i, y_i, z_i) is a point in C_i. But the piece of curve C_i can be approximated by a straight line segment L_i of length $\triangle s_i$, and the piece of the surface below the graph $z = f(x, y)$ and above the small piece of curve C_i in the xy plane is approximated by the vertical rectangle with base L_i and height $f(x_i, y_i, z_i$. The area of this rectangle is $f(x_i, y_i, z_i)\triangle s_i$. Hence the Riemann sum approximates the area of the surface.

Path Integrals. For an oriented curve C, write $\mathbf{s}$ for the position vector of a point on the curve, so

$$\mathbf{s}(x, y, z) \;=\; (x, y, z) \;=\; x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Write the formal expression

$$d\mathbf{s} \;=\; (dx, dy, dz) \;=\; dx\mathbf{i} \;+\; dy\mathbf{j} \;+\; dz\mathbf{k}.$$

If $\mathbf{F}(x, y, z)$ is a vector field defined at each point $(x, y, z) \in C$, write $\mathbf{F}$ in components $\mathbf{F} = (F_1, F_2, F_3)$. Then for some parameterization $\mathbf{c}$ of C, we define

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \;=\; \int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz$$

$$= \int_a^b \left(F_1(x(t), y(t), z(t))\frac{dx}{dt} \;+\; F_2(x(t), y(t), z(t))\frac{dy}{dt} \;+\; F_3(x(t), y(t), z(t))\frac{dz}{dt} \right) dt.$$

Example 2. Evaluate $\displaystyle\int_{\mathbf{c}} x^2 dx + xy dy + dz$ where $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.

Solution.

$$\int_0^1 \left(x^2 \frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt} \right) dt = \int_0^1 (t^2 + 2t^4) dt = \frac{1}{3}t^3 + \frac{2}{3}t^5 \Big|_0^1 = \frac{1}{3} + \frac{2}{5}.$$

Questions. 1. The curve in the previous example started at $(0, 0, 1)$ and ended at $(1, 1, 1)$. Suppose we take a new curve with the same endpoints. Will be get the same value for $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{s}$?

2. Suppose we reparameterize a curve reversing the orientation. Will $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{s}$ stay the same? How about if we do not reverse the orientation?

Example 3. To investigate the questions above, evaluate $\displaystyle\int_{\mathbf{c}} x^2 dx + xy dy + dz$ where $\mathbf{c}$ is the curve

(i). $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.
(ii). $\mathbf{c}(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.
(iii). $\mathbf{c}(t) = t^2\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.

Solution. (i).

$$\int_0^1 \left(x^2 \frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt} \right) dt = \int_0^1 (t^2 + t^2) dt = \frac{2}{3}t^3 \Big|_0^1 = \frac{2}{3}.$$

The line integral can have a different value although the endpoints of this curve are the same as in Example 2.
(ii).

$$\int_0^1 \Big(x^2\frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt}\Big)dt = \int_0^1 -((1-t)^2 + (1-t)^2)dt = \frac{2}{3}(1-t)^3\Big|_0^1 = -\frac{2}{3}.$$

The line integral is multiplied by -1 when the orientation is reversed.

Solution. (iv).

$$\int_0^1 \Big(x^2\frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt}\Big)dt = \int_0^1 (2t^5 + 2t^5)dt = \frac{4}{6}t^6\Big|_0^1 = \frac{2}{3}.$$

The line integral does not change with a new parameterization with the same orientation.
Theorem (Line integrals of gradient vector fields) If $\mathbf{c} : [a,b] \to \mathbf{R}^3$ then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Pf This follows from the chain rule $\frac{d}{dt}f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$.
Remark. The problem with the vector field $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + \mathbf{k}$ is that it is not the gradient of a scalar field. That is why $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ depends on the curve and not just the endpoints.