MATH 20E VECTOR CALCULUS

Lecture 16: Line Integrals continued.

How do we define

$$\int_C \mathbf{F} \cdot d\mathbf{s}?$$

Here, C is an oriented curve, $\mathbf{F}$ is a vector field defined along C. We choose points P_0, P_1, P_2, $\ldots$, P_m on the curve C so that P_0 is the the point at the beginning of C, P_m is the point at the end, and you travel through the points P_j in order as you trace out the curve in the direction of the orientation. Set $\triangle\mathbf{s}_j = \mathbf{s}_j - \mathbf{s}_{j-1}$ and form the Riemann sum

$$\sum_{j=1}^{m} \mathbf{F}(P_j) \cdot \triangle\mathbf{s}_j.$$

In the limit as $m \to \infty$ and $\max_j \triangle\mathbf{s}_j \to 0$, this sum converges to the integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

If we can choose a C^1 parameterization $\mathbf{c}(t)$ of C with $t \in [a,b]$ which agrees with the orientation of C, then we can choose points $a = t_0 < t_1 < \cdots < t_n = b$ so that $\mathbf{c}(t_j) = P_j$. Then when $\Delta t_j = t_j - t_{j-1}$ is small, we have

$$\triangle\mathbf{s}_j \;=\; \mathbf{s}_j - \mathbf{s}_{j-1} \;\approx\; \mathbf{c}'(t_j)(t_j - t_{j-1}) \;=\; \mathbf{c}'(t_j)\triangle t_j.$$

This approximation gets better and better as $\triangle t_j$ gets smaller and smaller. Hence

$$\sum_{j=1}^{m} \mathbf{F}(P_j) \cdot \triangle\mathbf{s}_j \;\approx\; \sum_{j=1}^{m} \mathbf{F}(\mathbf{c}(t_j))\triangle t_j \;\to\; \int_a^b \mathbf{F}(\mathbf{c}(t)) \;\cdot \mathbf{c}'(t)\, dt,$$

as $m \to \infty$ while $\max_j \triangle t_j \to 0$. This argument can be made rigorous and gives the formula

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \int_a^b \mathbf{F}(\mathbf{c}(t)) \;\cdot \mathbf{c}'(t)\, dt.$$

Example. Explain what happens to the integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

if the curve C is replaced by the curve $-C$ which is the same curve with the opposite orientation.

Solution.

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \lim_{\substack{m\to\infty \\ \max_j \|\triangle\mathbf{s}_j\|\to 0}} \sum_{j=1}^{m} \mathbf{F}(P_j) \cdot \triangle\mathbf{s}_j.$$

When the orientation is reversed, the points are listed in the opposite order so the change $\mathbf{s}_j$ changes sign. Indeed, using the points P_j which correspond to the orientation for C, we have that $\int_{-C} \mathbf{F} \cdot d\mathbf{s}$ is a limit of the Riemann sums

$$(*) \qquad \sum_{j=1}^{m} \mathbf{F}(P_{j-1}) \cdot (\mathbf{s}_{j-1} - \mathbf{s}_j) \;=\; -\sum_{j=1}^{m} \mathbf{F}(P_{j-1}) \cdot \triangle\mathbf{s}_j.$$

When the changes $\triangle\mathbf{s}_j$ get small, $\mathbf{F}(P_{j-1}) \approx \mathbf{F}(P_j)$, so (*) is approximately negative the Riemann sum which approximates $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Summary: When C is replaced by $-C$, $d\mathbf{s}$ changes to $-d\mathbf{s}$.

Remark: This is different from $\int_C f\, ds$. When C changes to $-C$, the sign of ds still remains positive, because arclength is defined so that it increases as you move along the curve in the direction of the orientation, regardless of what the orientation is.

Example 1. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$, and where C is the boundary of the region which is given by $x^2 + y^2 \le 1$, $x + y \ge 1$ and $x \ge 1/\sqrt{2}$ traversed clockwise.

Solution. We parameterize each piece of the curve, C_1 is parameterized by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \le t \le \pi/4$ and C_2 is parameterized by $\mathbf{c}_2(t) = (1/\sqrt{2}, t)$ for $1/\sqrt{2} \le t \le 1$ and C_3 is parameterized by $\mathbf{c}_3(t) = (t, 1-t)$ for $\sqrt{2} \le t \le 1$. Then $C = C_1 - C_2 + C_3$ and

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \int_{C_1 - C_2 + C_3} \mathbf{F} \cdot d\mathbf{s} \;=\; \int_{C_1} \mathbf{F} \cdot d\mathbf{s} \;-\; \int_{C_2} \mathbf{F} \cdot d\mathbf{s} \;+\; \int_{C_3} \mathbf{F} \cdot d\mathbf{s}.$$

Now

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} \;=\; \int_0^{\pi/4} \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt \;=\; \int_0^{\pi/4} -\sin^2 t + \cos^2 t\, dt \;=\; \frac{\sin 2t}{2}\bigg|_0^{\pi/4} \;=\; \frac{1}{2}.$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} \;=\; \int_{1/\sqrt{2}}^{1} \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt \;=\; \int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}\, dt \;=\; \frac{1}{\sqrt{2}} \;-\; \frac{1}{2}.$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} \;=\; \int_{1/\sqrt{2}}^{1} \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt \;=\; \int_{1/\sqrt{2}}^{1} (1 - 2t)\, dt \;=\; (t - t^2)\big|_{1/\sqrt{2}}^{1} \;=\; \frac{1}{\sqrt{2}} \;-\; 1$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \frac{1}{2} \;-\; \left(\frac{1}{\sqrt{2}} \;-\; \frac{1}{2} \right) \;+\; \frac{1}{\sqrt{2}} \;-\; 1 \;=\; 0.$$

Quicker Solution. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} = \nabla(xy)$, so

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \int_C \nabla f \cdot d\mathbf{s} \;=\; f(\text{terminal point of } C) \;-\; f(\text{initial point of } C) \;=\; 0,$$

since C is a closed curve. In addition, we note that each piece of the integral, for example $\int_{C_1} \mathbf{F} \cdot d\mathbf{s}$ can be computed as xy at the terminal point minus xy at the initial point!

Example. Last lecture we showed that the vector field $\mathbf{F} = (x^2, xy, 1)$ cannot be written as a gradient, because we saw that $\int_C \mathbf{F} \cdot d\mathbf{s}$ does depend on the curve, and not just the endpoints. Show directly that it cannot.

Solution. Try to solve the three equations

$$\frac{\partial f}{\partial x} = x^2 \tag{1}$$

$$\frac{\partial f}{\partial y} = xy \tag{2}$$

$$\frac{\partial f}{\partial z} = 1. \tag{3}$$

We can integrate (1). Remembering what partial differentiation means, we get the general solution

$$f \;=\; \frac{x^3}{3} \;+\; C(y,z).$$

Here, $C(y,z)$ is constant in x, but can depend on y and z. Plug this into (2). We get

$$\frac{\partial C(y,z)}{\partial y} \;=\; xy.$$

However, the left hand side does not change when x changes, whereas (except when $y = 0$) the right hand side changes when x changes. Hence the two sides cannot be equal and we obtain a contradiction.

A **Parameterized surface** is given in terms of two parameters

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad \text{or} \qquad \mathbf{T}(u,v) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

A particular example of a parameterized surface is a graph:

$$z = f(x,y), \qquad \text{or} \qquad \mathbf{T}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

Example. The sphere $x^2 + y^2 + z^2 = r^2$ can be parameterized using spherical coordinates:

$$x = r\sin\phi\cos\theta, \quad x = r\sin\phi\sin\theta, \quad z = r\cos\phi, \qquad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi$$

It can however, not be written as one graph, but one for the southern hemisphere $z = -\sqrt{r^2 - x^2 - y^2}$ and one for the northern hemisphere $z = \sqrt{r^2 - x^2 - y^2}$.

A surface is locally close to its tangent plane which is determined by its normal that we now will find. Another description of a surface is a level surface

$$h(x,y,z)=0, \qquad \text{if} \qquad \nabla h(x,y,z)\neq \mathbf{0}.$$

(The graph is a special case with $h(x,y,z){=}z{-}f(x,y)$.) In this case a normal is

$$\mathbf{N}=\nabla h$$

To find the unit normal to a parameterized surface recall that for a parameterized curve we found the tangent by differentiating with respect to the parameter. Here, $u\to \mathbf{T}(u,v_j)$, where v_j is kept constant and u vary, is a parameterized curve and

$$\mathbf{T}_u=\frac{\partial \mathbf{T}}{\partial u}=\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}$$

is tangent to this curve, and hence to the surface. Similarly the vector

$$\mathbf{T}_v=\frac{\partial \mathbf{T}}{\partial v}=\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}$$

is tangent to the curves $v\to \mathbf{T}(u_i,v)$, and hence to the surface. The tangent plane to the surface is spanned by $\mathbf{T}_u$ and $\mathbf{T}_v$ so a normal to the surface is given by

$$\mathbf{N}=\mathbf{T}_u\times\mathbf{T}_v$$

Example. Find the tangent plane to the surface $x=u\cos v$, $y=u\sin v$, $z=u$ at the point $(1,0,1)$.
Solution. $\mathbf{T}=u\cos v\,\mathbf{i}{+}u\sin v\,\mathbf{j}{+}u\,\mathbf{k}$ so $\mathbf{T}_u=\cos v\,\mathbf{i}{+}\sin v\,\mathbf{j}{+}\mathbf{k}$, $\mathbf{T}_v=-u\sin v\,\mathbf{i}+ u\cos v\,\mathbf{j}$.
$\mathbf{N}=\mathbf{T}_u\times\mathbf{T}_v=\cdots=-u\cos v\,\mathbf{i}-u\sin v\,\mathbf{j}+u\,\mathbf{k}$.

At the point $(1,0,1)$ we have $\mathbf{N}=(-1,0,1)$ and the tangent plane has equation

$$-x+z \;=\; 0.$$