**Lecture 16: Line Integrals continued.**

**Lecture 17: Area of a parameterized surface.**

A **Parameterized surface** is given in terms of two parameters

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad \text{or} \qquad \mathbf{T}(u, v) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

where $(u, v)$ is in some region $D$. Cut $D$ into small rectangles $R_{ij}$ with vertex $(u_i, v_j)$.

$$R_{ij} = \{(u, v);\, u_i \le u \le u_i + \triangle u, v_j \le v \le v_j + \triangle v\}.$$

For $i, j$ fixed, $u \to T(u, v_0)$ and $v \to T(u_0, v)$ are curves on the surface and hence

$$\mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \text{and} \quad \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

are **tangent vectors** to the surface. For $(u, v) \sim (u_i, v_j)$ the surface is close to its tangent plane at the point $(u_i, v_j)$ and $\mathbf{T}$ is close to the **linear approximation**:

$$\mathbf{T}(u, v) \;\approx\; \mathbf{L}_{ij}(u, v) \;=\; \mathbf{T}(u_i, v_j) + \mathbf{T}_u(u_i, v_j)(u - u_i) + \mathbf{T}_v(u_i, v_j)(v - v_j)$$

The image $S_{ij} = \mathbf{T}(R_{ij})$ is close to $\mathbf{L}_{ij}(R_{ij})$ which is a parallelogram with adjacent sides $\mathbf{T}_u \triangle u$ and $\mathbf{T}_v \triangle v$ so

$$\text{Area}\,(S_{ij}) \;\approx\; \|\mathbf{T}_u \times \mathbf{T}_v\|\, \triangle u \triangle v = \|\mathbf{T}_u \times \mathbf{T}_v\|\, \text{Area}\,(R_{ij})$$

Summing up over all small rectangles in the $u$-$v$ plane we get

$$\text{Area}\,(S) = \sum \text{Area}\,(S_{ij}) \;\approx\; \sum \|\mathbf{T}_u \times \mathbf{T}_v(u_i, v_j)\| \triangle u \triangle v$$

and in the limit as $\triangle u, \triangle v \to 0$ we get the formula for the **surface area** of a parameterized surface:

$$\text{Area}\,(S) = \iint \|\mathbf{T}_u \times \mathbf{T}_v(u, v)\|\, dudv$$

**Example**. Using the parameterization $(x, y, z) = (r\cos\theta, r\sin\theta, r)$ for $0 \le r \le 1$, $0 \le \theta \le 2\pi$, calculate the surface area of the cone $z^2 = x^2 + y^2$ with $0 \le z \le 1$..

**Solution**. We have

$$dS \;=\; \|\mathbf{T}_r \times \mathbf{T}_\theta\| dr d\theta \;=\; \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \right\| dr d\theta \;=\; \|(-r\cos\theta, -r\sin\theta, r)\| dr d\theta$$

$$= \sqrt{2} r dr d\theta.$$

Hence area is

$$A \;=\; \int_S dS \;=\; \int_0^1 \int_0^{2\pi} \sqrt{2} r\, dr d\theta \;=\; \sqrt{2}\pi.$$

**Surface area of a graph.** In the special case of a graph $z = f(x,y)$ we have $\mathbf{T}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k}$ we have $\mathbf{T}_x = \mathbf{i} + f_x(x,y)\,\mathbf{k}$, $\mathbf{T}_y = \mathbf{j} + f_y(x,y)\,\mathbf{k}$ and

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}$$

and

$$\|\mathbf{T}_x \times \mathbf{T}_y\| = \sqrt{1 + f_x^2 + f_y^2}$$

and hence we get the formula for the area of a graph:

$$\text{Area}\,(S) = \iint \sqrt{1 + f_x^2 + f_y^2}\; dxdy$$

There is however a simpler way to remember this formula. If the graph of $f$ is a plane $S$, let $\mathbf{n}$ be the unit normal to $S$. Choose $\mathbf{e} = \mathbf{n} \times \mathbf{k}$ so that $\mathbf{e}$ is a horizontal vector parallel to $S$. Choose $\mathbf{f} = \mathbf{e} \times \mathbf{f}$ so $\mathbf{f}$ is vector in the plane orthonormal to $\mathbf{e}$. Suppose $S_{ij}$ is a rectangle in $S$ corresponding to a rectangle $R_{ij}$ in the $xy$ plane, and suppose $S_{ij}$ has sides $E$ and $F$ parallel to vectors $\mathbf{e}$ and $\mathbf{f}$ respectively. The side $E$ is horizontal and so projects down to a side of $R_{ij}$ having the same length. However, the side $F$ is at an angle $\cos\gamma$ to the horizontal where $\gamma$ is the angle between $\mathbf{n}$ and $\mathbf{k}$. Hence it projects down to a side with $|\cos\gamma|$ times the length. Hence

$$\text{Area of } S_{ij} = \frac{1}{|\cos\gamma|} \times \text{ Area of } R_{ij}.$$

$$dS \;=\; \frac{1}{|\cos\gamma|} dxdy \;=\; \frac{1}{|\mathbf{n}\cdot\mathbf{k}|} dxdy.$$

Since locally the surface can be approximated by the tangent plane, this formula actually holds for a surface, where now $\mathbf{n}$ is the unit normal to the surface which may vary from point to point. We can check that this agrees with our previous formula. Indeed, the normal to the graph $z = f(x,y)$ is

$$\nabla(z - f(x,y)) \;=\; -f_x\mathbf{i} \;-\; f_x\mathbf{j} \;+\; \mathbf{k}.$$

Hence a unit normal is

$$\mathbf{n} \;=\; \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1}}, \qquad \frac{1}{|\mathbf{n}\cdot\mathbf{k}|} \;=\; \sqrt{f_x^2 + f_y^2 + 1}.$$

**Example**. Find the area of the part of the cone $S$ given by $z^2 = x^2 + y^2$ with $0 \le z \le 1$.
**Solution**. The surface is a graph and the angle between the surface and the $x - y$

plane is $\gamma = \pi/4$, since when say $y = 0$ its just $z = |x|$. Hence $\cos\gamma = 1/\sqrt{2}$. Alternatively one can calculate $|\mathbf{n}\cdot\mathbf{k}|$. Hence

$$\text{Area}(S) = \int_{x^2+y^2\le 1} \frac{dxdy}{|\cos\gamma|} = \sqrt{2}\int_{x^2+y^2\le 1} dxdy = \sqrt{2}\times \text{Area of unit disc} = \sqrt{2}\pi.$$

**Example.** Find the area of the sphere $S$ of radius $r$.
**Solution.** Using the parametrization $\mathbf{T} = r\sin\phi\cos\theta\,\mathbf{i} + r\sin\phi\sin\theta\,\mathbf{j} + r\cos\phi\,\mathbf{k}$ we get $\mathbf{T}_\phi = r\cos\phi\cos\theta\,\mathbf{i} + r\cos\phi\sin\theta\,\mathbf{j} - r\sin\phi\mathbf{k}$ and $\mathbf{T}_\theta = -r\sin\phi\sin\theta\mathbf{i} + r\sin\phi\cos\theta\mathbf{j}$;

$$\mathbf{T}_\phi\times\mathbf{T}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= r^2\sin^2\phi\cos\theta\,\mathbf{i} + r^2\sin^2\phi\sin\theta\,\mathbf{j} + r^2\sin\phi\cos\phi\,\mathbf{k}$$

and $|\mathbf{T}_\phi\times\mathbf{T}_\theta| = r^2|\sin\phi|\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi} = r^2\sin\phi$. Hence $\text{Area}\,(S) = \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\,d\phi d\theta = \int_0^{2\pi} -r^2\cos\phi\big|_0^\pi\,d\theta = \int_0^{2\pi} 2r^2\,d\theta = 4\pi r^2.$