Lecture 18: Suppose we want to find the total volume of water in the oceans of the earth. At each point of the surface the depth of the ocean is given by a function f. To measure the total volume we divide up the surface S into smaller surface areas $\triangle S_{ij}$, each of which is so small that we can think of it as approximately flat under which the depth of the ocean is approximately constant. The volume of water below $\triangle S_{ij}$ is approximately $f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$, where (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$. The total volume of water in the ocean is approximately $\sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$. We therefore define the **surface integral** of a function f over the surface S to be

$$\iint_S f\, dS = \lim_{\triangle S_{ij}\to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Suppose that S is a parameterized surface: $\mathbf{T}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$, where $(u,v) \in D$. Let $R_{ij} = \{(u,v);\, u_i \le u \le u_i + \triangle u,\, v_j \le v \le v_j + \triangle v\}$ be a small rectangle in the u-v plane and let S_{ij} be the image of R_{ij} under the map $(u,v) \to \mathbf{T}(u,v)$. Then the area of S_{ij} is approximately the area in of the parallelogram in the tangent plane spanned by the vectors $\mathbf{T}_u\triangle u$ and $\mathbf{T}_v\triangle v$:

$$\triangle S_{ij} \sim \big\|\mathbf{T}_u \times \mathbf{T}_v(u_i, v_j)\big\|\triangle u\triangle v$$

Substituting this we get a Riemann sum for a double integral in the u-v plane. We therefore define the surface integral of a function f over a surface S:

$$\iint_S f\, dS = \iint_D f\big(x(u,v), y(u,v), z(u,v)\big)\big\|\mathbf{T}_u \times \mathbf{T}_v(u,v)\big\|\, dudv$$

We can symbolically write

$$dS = \big\|\mathbf{T}_u \times \mathbf{T}_v(u,v)\big\|\, dudv$$

Example. Find $\iint_S z^2\, dS$, where $S = \{(x,y,z);\, x^2 + y^2 + z^2 = 1\}$ is the unit sphere.
Solution 1. A parametrization is $\mathbf{T}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$, and we showed before that

$$dS = \big\|\mathbf{T}_\phi \times \mathbf{T}_\theta(\phi,\theta)\big\| d\phi\, d\theta = \sin\phi\, d\phi\, d\theta$$

Hence

$$\iint_S z^2\, dS = \int_0^{2\pi}\!\!\int_0^{\pi} \cos^2\phi\, \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} -\frac{\cos^3\phi}{3}\Big|_0^{\pi} d\theta = \int_0^{2\pi} \frac{2}{3}\, d\theta = \frac{4\pi}{3}$$

Solution 2. The northern hemisphere is a graph over the disc $x^2 + y^2 \le 1$ in the xy plane. The unit normal to the sphere is $\mathbf{n} = (x, y, z)$ so

$$|\cos\gamma| \;=\; |\mathbf{n}\cdot\mathbf{k}| \;=\; z \;=\; \sqrt{1 - x^2 - y^2},$$

and

$$dS \;=\; \frac{1}{|\mathbf{n}\cdot\mathbf{k}|}dxdy \;=\; \frac{1}{\sqrt{1-x^2-y^2}}dxdy.$$

Since the integral of z^2 over the southern and northern hemispheres are the same

$$\int_S z^2\,dS = 2\int_{S_+} z^2\,dS = 2\iint_{x^2+y^2\le 1}(1-x^2-y^2)\frac{dxdy}{\sqrt{1-x^2-y^2}}$$
$$= 2\int_0^{2\pi}\int_0^1 (1-r^2)^{1/2} r\,d\theta = 2\int_0^{2\pi}\frac{1}{3}(1-r^2)^{3/2}\Big|_0^1\,d\theta = 2\int_0^{2\pi}\frac{1}{3}\,d\theta = \frac{4\pi}{3}$$

Ex. Find $\iint_S x\,dS$, where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.
Sol. The surface is a piece of a plane $ax+by+cz=d$ and putting in the 3 points we get $a=d$, $b=d$ and $c=d$, e.g. $a=b=c=d=1$. The surface is therefore given by $h(x,y,z) = x+y+z = 1$ and $(x,y) \in D = \{(x,y); x \ge 0, y \ge 0, x+y \le 1\}$. The normal to the surface is therefore $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. We have

$$dS = \frac{dxdy}{|\cos\gamma|} = \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \sqrt{3}dxdy$$

If we rewrite $D = \{(x,y); 0 \le x \le 1,\, 0 \le y \le 1-x\}$ we get

$$\iint_S x\,dS = \iint_D x\sqrt{3}\,dxdy = \int_0^1\int_0^{1-x} x\sqrt{3}\,dy\,dx = \int_0^1 xy\sqrt{3}\Big|_{y=0}^{1-x} dx = \int_0^1 x(1-x)\sqrt{3}\,dx = \frac{\sqrt{3}}{6}$$

There are a couple of alternative ways to express the surface area element:

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\| dudv = \sqrt{\|\mathbf{T}_u\|^2\,\|\mathbf{T}_v\|^2 - (\mathbf{T}_u\cdot\mathbf{T}_v)^2}\;dudv$$

and since $\mathbf{T} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we can also write (after some work)

$$dS = \|\mathbf{T}_u \times \mathbf{T}_v\| dudv = \sqrt{\Big(\frac{\partial(x,y)}{\partial(u,v)}\Big)^2 + \Big(\frac{\partial(y,z)}{\partial(u,v)}\Big)^2 + \Big(\frac{\partial(x,z)}{\partial(u,v)}\Big)^2}\;dudv$$

Example. Suppose that the graph of the function $y = f(x)$ with $a \le x \le b$ is rotated about the y-axis. Compute a formula for the surface area of the surface of revolution and interpret geometrically.

Solution. Pameterize by polar coordinates (τ, ψ) in the xz plane so $\mathbf{T}(\tau, \psi) = (x, y, z) = (\tau \cos\psi, f(\tau), \tau \sin\psi)$ with $a \leq \tau \leq b$ and $0 \leq \psi \leq 2\pi$. Then

$$\mathbf{T}_\tau \times \mathbf{T}_\psi \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\psi & f'(\tau) & \sin\psi \\ -\tau\sin\psi & 0 & \tau\cos\psi \end{vmatrix} \;=\; \tau(f'(\tau)\cos\tau\mathbf{i} \;+\; 1 \;+\; f'(\tau)\sin\psi\mathbf{k}).$$

Then

$$\|\mathbf{T}_\tau \times \mathbf{T}_\psi\| \;=\; |\tau|\sqrt{(f'(\tau))^2 + 1}.$$

Hence

$$\text{Area} = 2\pi \int_a^b |x|\sqrt{(f'(x))^2 + 1}\, dx$$