

MATH 20E VECTOR CALCULUS

Lecture 19: Surface Integrals of vector functions.

Recall for a C^1 oriented curve C , and \mathbf{F} a continuous vector field along C , we can define the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

Here, $\mathbf{s} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ represents the position vector of a point.

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

Now in terms of the arclength parameter s we have

$$d\mathbf{s} = \frac{d\mathbf{s}}{ds} ds = \mathbf{T} ds,$$

where \mathbf{T} is the unit tangent pointing in the direction of the orientation. So

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T} ds.$$

However, usually it is easier to compute the line integral directly as

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_1 dx + F_2 dy + F_3 dz$$

than to compute $d\mathbf{T}$ and $ds = (ds/dt)dt$. Physically, if \mathbf{F} represents the force on a particle, then $\mathbf{F} \cdot \mathbf{v}$ is the work done by the force as the particle moves through the vector \mathbf{v} . When the force is variable, the integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is the work done when the particle moves along the curve C .

A surface is called **closed** if it has no boundary.

The sphere is closed but the upper hemisphere is not since its boundary is the equator. A closed surface has an inside and an outside.

An **oriented surface** is a two-sided surface with one side specified as the **outside**. The **outward** normal points away from the outside of the surface.

A orientation can hence be specified by giving an outward normal everywhere.

The upper hemisphere is orientable.

A Möbius strip is not orientable since it only has one side.

A surface is called **regular** if it has a tangent plane everywhere. A parameterized surface is regular if $\mathbf{T}_u \times \mathbf{T}_v$ is nonvanishing everywhere.

A sphere is a regular surface but a cone is not regular at the tip.

For a C^1 oriented surface S with orientation given by the unit normal \mathbf{n} , we set

$$d\mathbf{S} = \mathbf{n} dS.$$

For a continuous vector field \mathbf{F} on S we define

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} dS.$$

Example 1. Calculate $\int_S x\mathbf{i} \cdot d\mathbf{S}$ where S is the unit sphere $x^2 + y^2 + z^2 = 1$ with continuous unit normal \mathbf{n} defined so $\mathbf{n}(1, 0, 0) = (1, 0, 0)$.

Solution. A unit normal is $\nabla(x^2 + y^2 + z^2) = (2x, 2y, 2z)$ so $\mathbf{n} = (x, y, z)$. Then

$$\int_S x\mathbf{i} \cdot d\mathbf{S} = \int_S x\mathbf{i} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) dS = \int_S x^2 dS$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^\pi (\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta - 2\cos^2\phi) \sin\phi d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi (\sin^2\phi - 2\cos^2\phi) \sin\phi d\phi d\theta = \int_0^{2\pi} \int_0^\pi (1 - 3\cos^2\phi) \sin\phi d\phi d\theta \\ &= \int_0^{2\pi} -\cos\phi + \cos^3\phi \Big|_{\phi=0}^\pi d\theta = 0 \end{aligned}$$

Example 2. Calculate the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$ out of the surface S of the cube $C = \{(x, y, z); 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

Sol. The Cube has six sides S_1 with $x = 0$, S_2 with $x = 1$, S_3 with $y = 0$, S_4 with $y = 1$, S_5 with $z = 0$ and S_6 with $z = 1$. On S_1 , the outward normal is $-\mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n} = (y\mathbf{j} - 2z\mathbf{k}) \cdot (-\mathbf{i}) = 0$, on S_2 , $\mathbf{F} \cdot \mathbf{n} = (\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}) \cdot \mathbf{i} = 1$, on S_3 , $\mathbf{F} \cdot \mathbf{n} = 0$, on S_4 , $\mathbf{F} \cdot \mathbf{n} = 1$, on S_5 , $\mathbf{F} \cdot \mathbf{n} = 0$, and on S_6 , $\mathbf{F} \cdot \mathbf{n} = -2$. Since the area of each side is one it follows that

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS + \dots + \iint_{S_6} \mathbf{F} \cdot \mathbf{n} dS = 0 + 1 + 0 + 1 + 0 - 2 = 0$$

$d\mathbf{S}$ for parameterized surfaces: Suppose $\mathbf{T} : D \rightarrow S$, $(u, v) \rightarrow (x, y, z)$ is a C^1 parameterization of the oriented surface S . A normal to the surface is

$$\mathbf{T}_u \times \mathbf{T}_v,$$

and we assume that this is a positive multiple of the unit normal \mathbf{n} giving the orientation. Then

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \mathbf{n} dS = \int_S \mathbf{F} \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \|\mathbf{T}_u \times \mathbf{T}_v\| dudv \\ &= \int_S \mathbf{F} \cdot \mathbf{T}_u \times \mathbf{T}_v dudv. \end{aligned}$$

Lecture 20: Flux.

Last time: Suppose $\mathbf{T} : D \rightarrow S$, $(u, v) \rightarrow (x, y, z)$ is a C^1 parameterization of the oriented surface S . A normal to the surface is

$$\mathbf{T}_u \times \mathbf{T}_v,$$

and we assume that this is a positive multiple of the unit normal \mathbf{n} giving the orientation. Then

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \mathbf{n} dS = \int_S \mathbf{F} \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \|\mathbf{T}_u \times \mathbf{T}_v\| dudv \\ &= \int_S \mathbf{F} \cdot \mathbf{T}_u \times \mathbf{T}_v dudv. \end{aligned}$$

Example 1. Let S be the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \leq z \leq 1$ oriented so that the unit normal \mathbf{n} points out from the region $x^2 + y^2 - z^2 \leq 1$. Use the parameterization $\mathbf{T}(u, v) = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} + v\mathbf{k}$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$ to calculate

$$\iint_S y\mathbf{j} \cdot d\mathbf{S}.$$

Furthermore, calculate \mathbf{n} and dS in terms of the parameters u, v .

Solution. $\mathbf{T}_u = (-\sin u - v \cos u)\mathbf{i} + (\cos u - v \sin u)\mathbf{j}$ and $\mathbf{T}_v = -\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$;

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u - v \cos u & \cos u - v \sin u & 0 \\ -\sin u & \cos u & 1 \end{bmatrix} = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} - v\mathbf{k}$$

We see that at $(u, v) = (0, 0)$ we have $(x, y, z) = (1, 0, 0)$ and $\mathbf{T}_u \times \mathbf{T}_v = (1, 0, 0)$. This points out from the region $x^2 + y^2 - z^2 \leq 1$, so \mathbf{T} agrees with the orientation of S . Then

$$\begin{aligned} \iint_S (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 y\mathbf{j} \cdot (\mathbf{T}_u \times \mathbf{T}_v) dvdu \\ &= \int_0^{2\pi} \int_0^1 (\sin u + v \cos u)(\sin u + v \cos u) dvdu \\ &= \int_0^{2\pi} \pi \int_0^1 \sin^2 u + v \sin(2u) + v^2 \cos^2 u dvdu = \frac{4\pi}{3}. \end{aligned}$$

Note that

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \frac{(\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} - v\mathbf{k}}{\sqrt{1 + v^2}},$$

$$dS = |\mathbf{T}_u \times \mathbf{T}_v| dudv = \sqrt{1 + 2v^2} dudv.$$

Example. Repeat the previous example by thinking of the surface S as a graph.

Solution. S is a graph over the region D in the xy plane given by $1 \leq x^2 + y^2 \leq 2$. The normal is

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} - z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

$$|\mathbf{n} \cdot \mathbf{k}| = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

so

$$dS = \frac{1}{|\mathbf{n} \cdot \mathbf{k}|} dxdy = \frac{\sqrt{x^2 + y^2 + z^2}}{z} dxdy,$$

$$y\mathbf{j} \cdot \mathbf{n} = -\frac{y^2}{\sqrt{x^2 + y^2 + z^2}},$$

so

$$y\mathbf{j} \cdot \mathbf{n} dS = -\frac{y^2}{z} dxdy.$$

Then

$$\begin{aligned} \iint_S y\mathbf{j} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_1^{\sqrt{2}} \frac{r^2 \sin^2 \theta}{\sqrt{r^2 - 1}} r dr d\theta = \int_0^{2\pi} \left(\int_1^{\sqrt{2}} r \sqrt{r^2 - 1} + \frac{r}{\sqrt{r^2 - 1}} \right) \sin^2 \theta dr d\theta \\ &= \int_0^{2\pi} \sin^2 \theta \left(\frac{1}{3}(r^2 - 1)^{3/2} + (r^2 - 1)^{1/2} \right) \Big|_1^{\sqrt{2}} dr d\theta = \frac{4\pi}{3}. \end{aligned}$$

If \mathbf{F} represents the flow of a fluid, and S is a C^1 oriented surface, then

$$\mathbf{F} = \mu \mathbf{v},$$

where μ is density and \mathbf{v} is velocity of fluid particles, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

is called the **flux of \mathbf{F} across S** . It is the rate in mass/unit time at which fluid crosses S in the direction given by the orientation.