MATH 20E VECTOR CALCULUS

Lecture 2: Review of 20C continued.

Midterms: Monday during lectures. Feb 6 and Mar 6. No calculators, notes or books. There will be an overflow room.

Definition. The **determinant** of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Definition. The **determinant** of a 3×3 matrix is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Definition. The **vector or cross product** of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The determinant on the right is not a determinant in the usual sense, because the matrix contains vectors, not just real numbers, but it is a useful way to remember the formula.

Geometric Interpretations. The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$ and its magnitude is $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta =$ area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Here, θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Moreover, $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ form a right handed system. In particular, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

The absolute value of the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ equals the area of the parallelogram in the plane spanned by $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

If $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = c_1, c_2, c_3)$, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}.$$

In particular, it's absolute value is the volume of the parallelopiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Example 1. $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 3\mathbf{k}$. Calculate the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Solution. The area is $\|\mathbf{a} \times \mathbf{b}\|$.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k},$$

so the area is $\sqrt{(-5)^2 + (-3)^2 + 2^2} = \sqrt{38}$.

Example 2. Calculate the parametric and standard equations of the plane through the point $(2, 3, 4)$ and parallel to the two vectors $(6, 4, 7)$ and $(-6, 7, -7)$.

Principle. If $\mathbf{R}_0 = (x_0, y_0, z_0)$ is a point in a plane P and $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are two vectors which are parallel to P (but not to each other), then the position vector of a general point $\mathbf{R} = (x, y, z)$ in the plane can be written as

$$\mathbf{R} = \mathbf{R}_0 + \lambda\mathbf{a} + \mu\mathbf{b},$$

for some constants λ and μ. Equivalently

$$(x, y, z) = (x_0, y_0, z_0) + \lambda(a_1, a_2, a_3) + \mu(b_1, b_2, b_3).$$

We remark that thinking of this as a map

$$(\lambda, \mu) \to (x, y, z)$$

we get a one-to-one map from the plane $\mathbb{R}^2$ to the plane P. To see that it is one-to-one, suppose that (λ, μ) and (λ', μ') are both mapped to the same point (x, y, z). Then

$$\mathbf{R}_0 + \lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{R}_0 + \lambda'\mathbf{a} + \mu'\mathbf{b}.$$

Subtracting, this gives

$$(\lambda - \lambda')\mathbf{a} = (\mu' - \mu)\mathbf{b}.$$

Since we are assuming that $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other, one cannot be a multiple of the other. Hence we must have $\lambda - \lambda' = \mu - \mu' = 0$, and so $(\lambda, \mu) = (\lambda', \mu')$.

Solution. The position vector of the general point (x, y, z) is

$$(x, y, z) = (2, 3, 4) + \lambda(1, 0, 2) + \mu(-1, 1, 3).$$

Principle. If $\mathbf{R}_0$ is the position vector of a point in the plane P and $\mathbf{N}$ is normal to the plane P, then the position vector of a general point $\mathbf{R}$ in the plane satisfies the equation

$$(\mathbf{R} - \mathbf{R}_0) \cdot \mathbf{N} = 0.$$

Solution. In our case, we are not given the normal **N**, but instead we are given two vectors parallel to the plane. To compute the standard formula we must first compute the normal.

$$\mathbf{N} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} \;=\; -2\mathbf{i} \;-\; 5\mathbf{j} \;+\; \mathbf{k}.$$

Now we get the equation

$$((x, y, z) \;-\; (2, 3, 4)) \;\cdot\; (-2, -5, 1) \;=\; 0.$$

This reduces to

$$-2x - 5y + z \;=\; 2 \cdot -2 \;+\; 3 \cdot -5 \;+\; 4 \cdot 1 \;=\; -15.$$

Matrix multiplication

Definition. An $m \times n$ matrix $A = [a_{ij}]$ is a collection of $m \cdot n$ numbers

$$\begin{matrix} & j \text{ th } \textbf{column} \\ i \text{ th } \textbf{row} & \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ & & \cdot & & \\ & & \cdot & & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ & & \cdot & & \\ & & \cdot & & \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \end{matrix}$$

A special case are the $1 \times n$ matrices which are row vector $(x_1, ..., x_n)$ or the $m \times 1$ matrices which are column vectors $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}$. Another important case are the square $n \times n$ matrices, e.g. a 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$ or a 3×3 matrix $\begin{pmatrix} 1 & 2 & 7 \\ 5 & 4 & 1 \\ 3 & 9 & 2 \end{pmatrix}$.

The **matrix multiplication** of an $m \times n$ matrix $A = [a_{ij}]$ by an $n \times k$ matrix $B = [b_{ij}]$ is the $m \times k$ matrix $C = [c_{ij}]$ whose entries are given by $c_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj}$. The c_{ij} entry is the dot product between the i th row of A and the j th column of B

$$\begin{pmatrix} & b_{1j} & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ & b_{nj} & \end{pmatrix} \begin{pmatrix} & & & & \\ a_{i1} & . & . & . & a_{in} \\ & & & & \end{pmatrix} \begin{pmatrix} & & \\ & c_{ij} & \\ & & \end{pmatrix}$$

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 & 1 \cdot 0 + 2(-1) \\ 3 \cdot 1 + 4 \cdot 2 & 3 \cdot 0 + 4(-1) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 11 & -1 \end{pmatrix}$

Any $m \times n$ matrix A determines a **mapping** $\mathbf{R}^n \to \mathbf{R}^m$ defined as follows.
Let $\mathbf{x} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathbf{R}^n$ and define

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \cdot & \\ & \cdot & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = A\mathbf{x}.$$

Example. Write the parametric equation of the plane through the origin: $(x, y, z) = \lambda(1, -2, -1) + \mu(4, 3, 1)$ in matrix form.

Solution. We write the equations as

$$\begin{aligned} x &= \lambda + 4\mu, \\ y &= -2\lambda + 3\mu, \\ z &= -\lambda + \mu. \end{aligned}$$

Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.$$