Lecture 20: Flux.

Last time: Suppose $\mathbf{T} : D \to S$, $(u, v) \to (x, y, z)$ is a C^1 parameterization of the oriented surface S. A normal to the surface is

$$\mathbf{T}_u \times \mathbf{T}_v,$$

and we assume that this is a positive multiple of the unit normal $\mathbf{n}$ giving the orientation. Then

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_S \mathbf{F} \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \|\mathbf{T}_u \times \mathbf{T}_v\|\, dudv \\ &= \int_S \mathbf{F} \cdot \mathbf{T}_u \times \mathbf{T}_v\, dudv.\end{aligned}$$

Example 1. Let S be the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \leq z \leq 1$ oriented so that the unit normal $\mathbf{n}$ points out from the region $x^2 + y^2 - z^2 \leq 1$. Use the parameterization $\mathbf{T}(u, v) = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} + v\mathbf{k}$, $\quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 1$ to calculate

$$\iint_S y\mathbf{j} \cdot d\mathbf{S}.$$

Furthermore, calculate $\mathbf{n}$ and dS in terms of the parameters u, v.

Solution. $\mathbf{T}_u = (-\sin u - v \cos u)\mathbf{i} + (\cos u - v \sin u)\mathbf{j}$ and $\mathbf{T}_v = -\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$;

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u - v \cos u & \cos u - v \sin u & 0 \\ -\sin u & \cos u & 1 \end{bmatrix} = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} - v\mathbf{k}$$

We see that at $(u, v) = (0, 0)$ we have $(x, y, z) = (1, 0, 0)$ and $\mathbf{T}_u \times \mathbf{T}_v = (1, 0, 0)$. This points out from the region $x^2 + y^2 - z^2 \leq 1$, so $\mathbf{T}$ agrees with the orientation of S. Then

$$\begin{aligned}\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 y\mathbf{j} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, dvdu \\ &= \int_0^{2\pi} \int_0^1 (\sin u + v \cos u)(\sin u + v \cos u)\, dvdu \\ &= \int_0^2 \pi \int_0^1 \sin^2 u + v \sin(2u) + v^2 \cos^2 u\, dvdu = \frac{4\pi}{3}.\end{aligned}$$

Note that

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \frac{(\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} - v\mathbf{k}}{\sqrt{1 + v^2}},$$

$dS = |\mathbf{T}_u \times \mathbf{T}_v|\, dudv = \sqrt{1+2v^2}\, dudv.$

Example. Repeat the previous example by thinking of the surface S as a graph.

Solution. S is a graph over the region D in the xy plane given by $1 \le x^2+y^2 \le 2$. The normal is

$$\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}-z\mathbf{k}}{\sqrt{x^2+y^2+z^2}},$$

$$|\mathbf{n}\cdot\mathbf{k}| = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

so

$$dS = \frac{1}{|\mathbf{n}\cdot\mathbf{k}|} dxdy = \frac{\sqrt{x^2+y^2+z^2}}{z}\, dxdy,$$

$$y\mathbf{j}\cdot\mathbf{n} = -\frac{y^2}{\sqrt{x^2+y^2+z^2}},$$

so

$$y\mathbf{j}\cdot\mathbf{n}\, dS = -\frac{y^2}{z}\, dxdy.$$

Then

$$\iint_S y\mathbf{j}\cdot d\mathbf{S} = \int_0^{2\pi}\int_1^{\sqrt{2}} \frac{r^2\sin^2\theta}{\sqrt{r^2-1}}\, rdrd\theta = \int_0^{2\pi}\left(\int_1^{\sqrt{2}} r\sqrt{r^2-1} + \frac{r}{\sqrt{r^2-1}}\right)\sin^2\theta\, drd\theta$$
$$= \int_0^{2\pi}\sin^2\theta\left(\frac{1}{3}(r^2-1)^{3/2}+(r^2-1)^{1/2}\right)\Big|_1^{\sqrt{2}} drd\theta = \frac{4\pi}{3}.$$

If $\mathbf{F}$ represents the flow of a fluid, and S is a C^1 oriented surface, then

$$\mathbf{F} = \mu\mathbf{v},$$

where μ is density and $\mathbf{v}$ is velocity of fluid particles, then

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

is called the **flux of F across** S. It is the rate in mass/unit time at which fluid crosses S in the direction given by the orientation.