MATH 20E VECTOR CALCULUS

Lecture 21: Flux.

$\mathbf{F}$ represents the **flow** of a fluid (flow rate density).

$$\mathbf{F} = \mu \mathbf{v},$$

where μ is the density and $\mathbf{v}$ is the velocity.

If S is a small piece of (oriented) plane with area $\triangle S$ and unit normal $\mathbf{n}$ then we claim that the rate (in mass/unit time) that fluid crosses S from the "inside" to the "outside" (by definition, the outside is the side that $\mathbf{n}$ is pointing to) is given by

$$\mathbf{F} \cdot \mathbf{n} \, \triangle S$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{v}\triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{v} \cdot \mathbf{n} \, \triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu \mathbf{v} \cdot \mathbf{n} \, \triangle t \triangle S$.

Now let $\mathbf{F}$ be the flow of a fluid and let S be a C^1 oriented closed surface. We divide S into tiny pieces S_{ij}, having surface area $\triangle S_{ij}$. Let P_{ij} be a point in S_{ij}. If the piece of surface S_{ij} is very small, then it is close to a piece of the tangent plane to S at P_{ij}. In time $\triangle t$, the mass of fluid crossing S_{ij} from the inside to the outside is approximately

$$\mathbf{F}(P_{ij}) \cdot \mathbf{n}(P_{ij}) \, \triangle S_{ij} \triangle t.$$

Summing over the pieces of surface, we find that the mass of fluid crossing S in time $\triangle t$ is approximately

$$\sum_{i,j} \mathbf{F}(P_{ij}) \cdot \mathbf{n}(P_{ij}) \, \triangle S_{ij} \triangle t.$$

As we compute this with more and more pieces with the maximum diameter tending to zero, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS \, \triangle t.$$

This represents the total mass of fluid crossing S from the "inside" to the "outside" in time $\triangle t$. Strictly speaking it is the mass of fluid crossing in the "correct" direction, minus the mass of fluid crossing in the "wrong" direction. The flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

represents the rate (in mass/unit time) at which fluid crosses over from the "inside" of S to the "outside".

Example. Let R be the 3-dimensional region $R = \{x^2 + z^2 \leq 4(1 - y),\, y \geq 0\}$. Let S be the surface of R with the normal oriented outwards. Note that S has two parts $\{x^2 + z^2 = 4(1 - y),\, y \geq 0\}$ and $\{y = 0,\, x^2 + z^2 \leq 4\}$.
a) Find the area of S.
b) Find the flux of $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + \mathbf{k}$ through S.

Solution. Let S_1 be the flat part of surface $\{y = 0,\, x^2 + z^2 \leq 4\}$. Let S_2 be the curved part $\{x^2 + z^2 \leq 4(1 - y),\, 0 \leq y \leq 1\}$. The area of S_1 is 4π. S_2 can be viewed as a graph $y = g(x, z) = 1 - (x^2 + z^2)/4$ over the disc $D = \{x^2 + z^2 \leq 4\}$ in the xz-plane. A normal is given by $\nabla(x^2 + y^2 + 4(y - 1)) = 2x\mathbf{i} + 4\mathbf{j} + 2z\mathbf{k}$. The outward unit normal is

$$\mathbf{n} \;=\; \frac{x\mathbf{i} + 2\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + z^2 + 4}}.$$

Now

$$dS \;=\; \frac{dxdz}{|\mathbf{n} \cdot \mathbf{j}|} \;=\; \frac{\sqrt{x^2 + z^2 + 4}}{2}\, dxdz.$$

Introducing polar coordinates in the xz-plane:

$$\iint_{S_2} dS \iint_D \frac{\sqrt{x^2 + z^2 + 4}}{2} dxdz \;=\; \int_0^2 \int_0^{2\pi} \frac{\sqrt{r^2 + 4}}{2} r d\theta r dr$$

$$= \frac{2\pi}{6}(r^2 + 4)^{3/2}\Big|_0^2 \;=\; \frac{8\pi}{3}(2^{3/2} - 1)$$

(b) The normal to S_1 is $\mathbf{n} = -\mathbf{j}$ and there $\mathbf{F} \cdot \mathbf{n} = y = 0$ so the integral over S_1 vanishes. On S_2,

$$\mathbf{F} \cdot \mathbf{n} \;=\; \frac{x^2 - 2y + z}{\sqrt{x^2 + y^2 + 4}},$$

We obtain

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \frac{x^2 - 2y + z}{2} dxdz \;=\; \iint_D \left(x^2 + \frac{x^2 + z^2}{2} - 2 + z\right))\, dxdz.$$

Now $\iint_D z\, dxdz$ vanishes by symmetry, and $\iint_D z^2\, dxdz = \int_D x^2 dxdz$ by symmetry. Introducing polar coordinates in the xz plane, we are left with

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, ds \;=\; 2 \iint_D (x^2 - 1)\, dxdz \;=\; 2\left(\int_0^2 \int_0^{2\pi} 2r^3 \cos^2\theta\, d\theta dr \;-\; 4\pi\right)$$

$$= 2(4\pi - 4\pi) \;=\; 0.$$

Alternative solution: $\nabla \cdot \mathbf{F} = 0$. Since we are computing the flux over the boundary of a solid region, the answer should be zero. We will understand this in the next two weeks!