Lecture 22: Green's theorem. Recall the first fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

Suppose that D is a domain in the plane with piecewise C^1 boundary curve C. We say that C is *oriented in positive direction* if walking in the direction of C the domain D is on your left, so your right arm gives the outward unit normal. (If D is an elementary region then C then the positive orientation is *anticlockwise*.)
Green's theorem. If P and Q are C^1 on D then

$$\int_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx dy$$

Example. Let $P = xy$, $Q = y^2$ and $D = \{(x, y);\, 0 \le x \le 1,\, x^2 \le y \le x\}$. Let C be the positively oriented boundary of D. Calculate both sides of Green's theorem.
Solution. The boundary consists of two parts C_1; $x = t$, $y = t^2$, $0 \le t \le 1$ and C_2; $x = 1-t$, $y = 1-t$, $0 \le t \le 1$. Note that C_2 is oriented so it starts at $(x, y) = (1, 1)$ in order that the total curve should be positively oriented. Let $P = xy$ and $Q = y^2$.

$$\int_C P\,dx + Q\,dy = \int_{C_1} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\,dt + \int_{C_2} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\,dt$$

$$= \int_0^1 t^3 + 2t^5\,dt + \int_0^1 -(1-t)^2 - (1-t)^2\,dt = \frac{t^4}{4} + \frac{t^6}{3}\Big|_0^1 + \frac{2(1-t)^3}{3}\Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{2}{3} = -\frac{1}{12}$$

On the other hand

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dxdy = \iint_D -x\,dxdy = \int_0^1 \int_{x^2}^x -x\,dy\,dx = \int_0^1 -x(x - x^2)\,dx$$

$$= -\frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$$

Using Green's theorem to compute area. Apply Green's Theorem to $(Q, P) = (x, 0)$ and $(Q, P) = (0, -y)$ we get

$$\text{Area}\,(D) = \iint_D 1\,dxdy = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C -y\,dx + x\,dy$$

which gives another way to calculate the area.
Example. Find the area of the interior of an ellipse: $D = \{(x, y);\, (x/a)^2 + (y/b)^2 \le 1\}$
Solution. Parameterizing the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$,

$$\text{Area}\,(D) = \frac{1}{2}\int_C x\,dy = \frac{1}{2}\int_0^{2\pi} x\frac{dy}{dt}\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)\frac{d(b\sin t)}{dt}\,dt = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t\,dt = \pi ab$$

Different ways to write Green's Theorem.
1. Split into two equations

$$\int_C P\,dx \;=\; -\iint_D \frac{\partial P}{\partial y}\,dxdy, \qquad\qquad \int_C Q dy = \iint_D \frac{\partial Q}{\partial x} dx.$$

2. Curl formulation: $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$,

$$\int_C \mathbf{F}\cdot d\mathbf{s} \;=\; \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA.$$

Indeed, it is a homework problem to check

$$\nabla\times\mathbf{F} \;=\; \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

3. Divergence formulation: $\mathbf{F} = Q\mathbf{i} - P\mathbf{j}$, and $\mathbf{n}$ is the outward unit normal.

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds \;=\; \iint_D \nabla\cdot\mathbf{F}\,dA.$$

Indeed,

$$\nabla\cdot\mathbf{F} \;=\; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

But

$$P\,dx \;+\; Q\,dy \;=\; \mathbf{F}\cdot(dy\mathbf{i} - dx\mathbf{j})\,,$$

and

$$(dy\mathbf{i} - dx\mathbf{j}) \;=\; \left(\frac{dy}{ds}\mathbf{i} \;-\; \frac{dx}{ds}\mathbf{j}\right)\,ds \;=\; \mathbf{n}\,ds,$$

where $\mathbf{n}$ is the outward unit normal.

Proof of Green's theorem. We prove

$$\int_C Pdx = \iint_D -\frac{\partial P}{\partial y}dxdy, \qquad \text{and} \qquad \int_C Qdy = \iint_D \frac{\partial Q}{\partial x}dxdy.$$

Let us just prove the first since the proof for the second is similar. Suppose D is y-simple: $D = \{(x,y);\, a \le x \le b,\ f_1(x) \le y \le f_2(x)\}$. By the fund. th.

$$-\iint_D \frac{\partial P}{\partial y}dxdy = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}dy\,dx = \int_a^b -P(x, f_2(x))\,dx + \int_a^b P(x, f_1(x))\,dx.$$

The boundary has two parts where $y = f_1(x)$ and $y = f_2(x)$ and the two integrals are exactly the line integrals on these curves positively oriented.

In general, split D into simple regions.

Example. Calculate

$$\int_C \frac{-y}{x^2+y^2}\,dx \;+\; \frac{x}{x^2+y^2}\,dy,$$

where C is the boundary of the unit square $-2 \le x \le 2$ and $-2 \le y \le 2$.

Solution. It would be easier to integrate over a circle. Consider the region D given by the square minus the disc $x^2+y^2 \le 1$. Its boundary is $C - C'$ where C' is the circle $x^2+y^2=1$ oriented anticlockwise. Now

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \;=\; \frac{\partial}{\partial x}\frac{x}{x^2+y^2} \;-\; \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} \;=\; 0.$$

Hence by Green's Theorem

$$\int_C P dx + Q dy \;-\; \int_{C'} P dx + Q dy \;=\; 0,$$

and

$$\int_C P dx + Q dy \;=\; \int_{C'} P dx + Q dy \;=\; \int_{C'} -y dx + x dy \;=\; 2 \text{ area disc} \;=\; 2\pi.$$