# MATH 20E VECTOR CALCULUS

**Lecture 23: Stokes' theorem.** Let $S$ be an oriented surface with unit normal $\mathbf{n}$ and boundary $C$. Then $C$ can be given an orientation *induced by* $\mathbf{n}$ (or *compatible with that of* $S$). Stand on the surface $S$ near $C$ with your head pointing towards $\mathbf{n}$ so that $S$ to your left and your right arm pointing out of $S$. Then you are facing in the direction of $C$.

**Stokes' theorem**. If $S$ is a piecewise $C^1$ surface with piecewise $C^1$ boundary, and $\mathbf{F}$ is a $C^1$ vector field on $S$, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$$

If $S$ is a domain in the $x$-$y$ plane then Stoke's theorem reduces to Green's theorem. In fact $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy$, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$, if $\mathbf{n} = \mathbf{k}$.

**Example**. Calculate both sides of Stoke's theorem when $S$ is the upper hemisphere given by

$$(x, y, z) \in S \;\Leftrightarrow\; x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0,$$

oriented with an upward pointing unit normal $\mathbf{n}$, *i.e.* $\mathbf{n} \cdot \mathbf{k} > 0$, and with

$$\mathbf{F} \;=\; \frac{-y}{x^2 + y^2 + z^2}\mathbf{i} \;+\; \frac{x}{x^2 + y^2 + z^2}\mathbf{j}.$$

**Solution**. The boundary of $S$ is the circle $x^2 + y^2 = 1$ in the $xy$ plane oriented anticlockwise. On the unit circle, $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Hence parameterizing $(x, y, z) = (\cos t, \sin t, 0)$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; \int_C -y dx + x dy \;=\; \int_0^{2\pi} -\sin t(-\sin t) + \cos t(\cos t)\, dt \;=\; \int_0^{2\pi} 1\, dt \;=\; 2\pi.$$

On the other hand, we calculate

$$\nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2+z^2} & \frac{x}{x^2+y^2+z^2} & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\frac{x}{x^2 + y^2 + z^2}\mathbf{i} - \frac{\partial}{\partial z}\frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{x}{x^2 + y^2 + z^2} \;-\; \frac{\partial}{\partial y}\frac{y}{x^2 + y^2 + z^2}\right)\mathbf{k}$$

$$= \frac{2xz}{(x^2 + y^2 + z^2)^2}\mathbf{i} \;+\; \frac{2yz}{(x^2 + y^2 + z^2)^2}\mathbf{j} \;+\; \frac{2z^2}{(x^2 + y^2 + z^2)^2}\mathbf{k}$$

$$= \frac{2z}{(x^2 + y^2 + z^2)^2}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,.$$

On the surface $S$ we have $x^2+y^2+z^2=1$ and so $\nabla \times \mathbf{F} = 2z(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$. Since $\mathbf{n} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, we have $\nabla \times \mathbf{F} = 2z$. Then using spherical coordinates,

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS \;=\; \int_0^{2\pi} \int_0^{\pi/2} 2\cos\phi\, \sin\phi\, d\phi d\theta \;=\; \int_0^{2} \pi \sin^2\phi\big|_0^{\pi/2}\, d\theta = \; 2\pi.$$

**Remark**. There are of course other surfaces whose boundary is the circle $C$. For example we can take the disc $D$ given by $z=0$ and $x^2+y^2 \le 1$. On this disc we have $\nabla \times \mathbf{F} = 0$. This would seem to contradict Stoke's theorem, since then

$$\iint_D \nabla \times \mathbf{F} \cdot \mathbf{n}\, dxdy \;=\; 0.$$

while we already computed

$$\int_C \mathbf{F} \cdot d\mathbf{s} \;=\; 2\pi.$$

However, this apparent contradiction is explained by the fact that the vector field $\mathbf{F}$ has a singularity at the point $(0,0,0)$ which is contained in $D$. To apply Stoke's theorem we need $\mathbf{F}$ to be $C^1$ on $D$, which it is not. We notice that the hemisphere $S$ does not contain the origin, so Stoke's theorem does apply to $S$.

**Geometric Interpretation of Curl**. Let $\mathbf{F}$ represent the flow of a fluid. If you place a small paddle wheel in the fluid at the point $(x,y,z)$ and rotate its axis you find that when the axis of the paddle wheel points in the direction of $\nabla \times \mathbf{F}$, the paddle wheel turns fastest. It turns in the direction given by the right hand rule. This can be explained by Stoke's theorem. Take a small planar disc $S$ with normal $\mathbf{n}$, center $(x,y,z)$, and boundary circle $C$. Then if the disc is really small,

$$(\nabla \times \mathbf{F})(x,y,z) \cdot \mathbf{n} \;\approx\; \frac{1}{\text{area } S} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS.$$

By Stoke's theorem, this equals

$$\frac{1}{\text{area } S} \int_C \mathbf{F} \cdot d\mathbf{s}.$$

But recall that if the tangent to the curve $S$ is $\mathbf{t}$, then

$$\int_C \mathbf{F} \cdot \mathbf{s} \;=\; \int_C \mathbf{F} \cdot \mathbf{t}\, ds.$$

This is measuring the extent to which $\mathbf{F}$ swirls around the curve $C$ in the direction of the orientation, and hence describes how fast the paddle wheel will turn.

**Example.** Find the integral $\int_C -y^3dx + x^3dy - z^3dz$, where $C$ is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$ and the orientation of $C$ corresponds to a counterclockwise motion in the $x$-$y$ plane.

**Solution. 1.** Let $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$. The integral is by Stokes Theorem equal to the surface integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over some surface $S$ with the boundary $C$ and with unit normal positively oriented with respect to the orientation of the boundary. We have $\nabla \times \mathbf{F} = ... = (3x^2 + 3y^2)\mathbf{k}$. We take $S$ to be the region in the plane $h(x, y, z) = x + y + z = 1$ with boundary $C$. A unit normal to $S$ is given by $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ and it has the correct orientation since $\mathbf{n} \cdot \mathbf{k} = 1/\sqrt{3} > 0$. We therefore get

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S 3(x^2 + y^2)/\sqrt{3}\, dS$$

Writing $dS = dxdy/|\mathbf{n} \cdot \mathbf{k}| = \sqrt{3}dxdy$ we get

$$\iint_{x^2+y^2\leq 1} 3(x^2 + y^2)\, dxdy = \int_0^{2\pi}\int_0^1 3r^2\, rdr\, d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_0^1 d\theta = 2\pi\frac{3}{4} = \frac{3\pi}{2}$$

**Sol. 2.** Directly calculating the line integral. Parameterizing the curve $C$ we can write $x = \cos t$, $y = \sin t$ and $z = 1 - x - y = 1 - \cos t - \sin t$, $0 \leq t \leq 2\pi$ and write

$$\begin{aligned}\int_C -y^3 dx + x^3 dy - z^3 dz &= \int_0^{2\pi} \Big( - y^3\frac{dx}{dt} + x^3\frac{dy}{dt} - z^3\frac{dz}{dt}\Big)\, dt \\ &= \int_0^{2\pi} \big(\sin^4 t + \cos^4 t + (1 - \cos t - \sin t)^3(\sin t - \cos t)\, dt\end{aligned}$$

But this is to much work to calculate.

**Proof of Stokes' theorem.** for a graph $z = f(x, y)$, $(x, y) \in D$. Since the surface can be written $h(x, y, z) = z - f(x, y)$ a normal is given by $\mathbf{N} = \nabla h = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$ and the unit normal is given by $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. The surface measure is $dS = dxdy/|\mathbf{k} \cdot \mathbf{n}|$, where $\mathbf{k} \cdot \mathbf{n} = \mathbf{k} \cdot \mathbf{N}/|\mathbf{N}| = 1/|\mathbf{N}|$, so $dS = |\mathbf{N}|\, dxdy$ and hence

$$\iint_S \mathbf{G} \cdot \mathbf{n}\, dS = \iint_D -G_1 f_x - G_2 f_y + G_3\, dxdy, \qquad \text{if} \quad \mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$$

If we apply to $\mathbf{F}$ this to $\mathbf{G} = \nabla \times \mathbf{F}$ we get

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dxdy,$$

If we parameterize the boundary $x = x(t)$, $y = y(t)$ and $z = f(x, y)$ we have

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt},$$

by the chain rule, and

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b \Big(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\Big)\, dt = \int_a^b \Big(\big(F_1 + f_x F_3\big)\frac{dx}{dt} + \big(F_2 + f_y F_3\big)\frac{dy}{dt}\Big)\, dt$$

This can now be considered as a line integral in the plane:

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\partial D} P dx + Q\, dy, \qquad \text{where}$$

$$\begin{aligned} P(x,y) &= F_1(x,y,f(x,y)) + f_x(x,y)F_3(x,y,f(x,y)),\\ Q(x,y) &= F_2(x,y,f(x,y)) + f_y(x,y)F_3(x,y,f(x,y)) \end{aligned}$$

We can therefore apply Greens formula in the plane.

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} f_y + f_x \frac{\partial F_3}{\partial y} + f_x f_y \frac{\partial F_3}{\partial z} + f_{xy} F_3\\ \frac{\partial Q}{\partial x} &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} f_x + f_y \frac{\partial F_3}{\partial x} + f_x f_y \frac{\partial F_3}{\partial z} + f_{xy} F_3 \end{aligned}$$

so by Green's theorem

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\partial D} P dx + Q\, dy &= \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dxdy\\ &= \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dxdy, \end{aligned}$$

**Ex.** Show that $\int_C ye^z dx + xe^z dy + xye^z dz = 0$ for a closed curve $C$.
**Sol.** $\mathbf{F} = \nabla(xye^z)$ so $\nabla\times\mathbf{F} = 0$ and by Stokes's theorem the integral vanishes.