

**Lecture 24: Conservative Fields.**

Let  $\mathbf{F}$  be a  $C^1$  vector field defined on  $U = \mathbb{R}^n \setminus B$ . (We remove the *bad* set  $B$  from  $\mathbb{R}^n$  to allow  $\mathbf{F}$  to blow up or have singularities. We think of  $U$  as being the set of *good points*.)

**Definition.** The vector field  $\mathbf{F}$  is a *gradient* on the domain  $U = \mathbb{R}^n \setminus B$  if  $\mathbf{F} = \nabla f$  for some scalar field  $f$  defined on  $U$ .

**Theorem.** The following are equivalent.

- (a).  $\mathbf{F}$  is a gradient on  $U$ .
- (b).  $\int_C \mathbf{F} \cdot ds = 0$  for all closed oriented curves  $C$  contained in  $U$ .
- (c).  $\int_{C_1} \mathbf{F} \cdot ds = \int_{C_2} \mathbf{F} \cdot ds$  for all oriented curves  $C_1$  and  $C_2$  contained in  $U$  which have the same beginning point and the same terminal point.

When these conditions are satisfied, we say that the vector field  $\mathbf{F}$  is **conservative** on  $U$ .

(a)  $\Rightarrow$  (b):  $\int_C \mathbf{F} \cdot ds = \int_C \nabla f \cdot ds = f(\text{initial point of } C) - f(\text{terminal point of } C) = 0$ .

(b)  $\Rightarrow$  (c): If  $C_1$  and  $C_2$  have the same endpoints, then  $C_1 - C_2$  is a closed curve, so from (b),

$$0 = \int_{C_1 - C_2} \mathbf{F} \cdot ds = \int_{C_1} \mathbf{F} \cdot ds - \int_{C_2} \mathbf{F} \cdot ds.$$

(c)  $\Rightarrow$  (a): Choose a point  $P_0$  in  $U$  and for  $P \in U$  define  $f(P) = \int_C \mathbf{F} \cdot ds$  where  $C$  is any curve joining  $P_0$  to  $P$ . The precise curve does not matter because any two choices give the same answer by (c). Now we just show that  $\nabla f = \mathbf{F}$ . However,

$$f(x+h, y, z) - f(x, y, z) = \int_L \mathbf{F} \cdot ds = \int_0^h \mathbf{F}_1(x+t, y, z) dt.$$

where  $L$  is the line segment joining  $(x, y, z)$  to  $(x+h, y, z)$  so by the fundamental theorem of calculus,

$$\left. \frac{d}{dh} \right|_{h=0} f(x+h, y, z) = F_1(x, y, z).$$

A similar argument shows that  $f_y = F_2$  and  $f_z = F_3$ .

**Example.** Compute  $\int_C \mathbf{F} \cdot ds$  when  $C$  is the positively oriented unit circle  $x^2 + y^2 = 1$  in the  $xy$  plane and (a).  $\mathbf{F} = y\mathbf{i}$ , (b).  $\mathbf{F} = y\mathbf{i} + (x+y)\mathbf{j}$ . Determine whether  $\mathbf{F}$  is a gradient on  $\mathbb{R}^2$  in each case.

**Solution.** (a).  $\int_0^{2\pi} -\sin^2 t dt = \pi \neq 0$  so  $\mathbf{F}$  is not a gradient. (b).  $\int_0^{2\pi} -\sin^2 t + \cos^2 t + \sin t \cos t dt = 0$ . We cannot conclude that  $\mathbf{F}$  is a gradient, because in order

to be a gradient we need  $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$  for **every** closed curve  $C$ , not just for one curve. However, we can indeed solve

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + y, \quad \frac{\partial f}{\partial z} = 0.$$

We get from the first equation  $f = xy + C(y, z)$  and from the second equation  $C_y(y, z) = y$ , so  $C(y, z) = y^2/2 + C_2(z)$ . But from the third equation we get  $C_2'(z) = 0$  so  $C_2(z) = C_2$ , a constant independent of  $z$ . Hence the general solution is  $f = xy + y^2/2 + C_2$ , and we have  $\mathbf{F} = \nabla(xy + y^2/3)$ .

**Theorem.** If  $\mathbf{F}$  is conservative on  $U$  then  $\nabla \times \mathbf{F} = 0$  on  $U$ .

**Proof.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x + y & 0 \end{vmatrix} = \mathbf{0}.$$

**Question.** Is  $\nabla \times \mathbf{F} = 0$  equivalent to  $\mathbf{F}$  being conservative?

**Stokes' theorem.** If  $S$  is a piecewise  $C^1$  surface with unit normal  $\mathbf{n}$  and piecewise  $C^1$  boundary oriented by  $\mathbf{n}$ , and  $\mathbf{F}$  is a  $C^1$  vector field on  $S$ , then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS.$$

We see that we should be asking the question: Is a closed curve  $C$  in  $U$  always the boundary of a surface  $S$  in  $U$ ?

**Definition** The region  $U$  is **Simply Connected** if every closed curve  $C$  in  $U$  can be shrunk to a point while remaining in  $U$ . In this case, every closed curve in  $U$  is the boundary of a parameterized surface in  $U$ .

**Theorem.** If  $\nabla \times \mathbf{F} = 0$  on the simply connected region  $D$  then  $\mathbf{F}$  is conservative on  $D$ .

**Examples.** Which domains are simply connected?

- (a).  $\mathbb{R}^3$ ? Yes
- (b).  $\mathbb{R}^2$ ? Yes
- (c).  $\mathbb{R}^2 \setminus \{(0, 0)\}$ ? No
- (d).  $\mathbb{R}^2 \setminus \{(0, 0, 0)\}$ ? Yes
- (e).  $\mathbb{R}^3 \setminus z$ -axis? No

**Example.**

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

Calculate  $\int_C \mathbf{F} \cdot d\mathbf{s}$  and  $\nabla \times \mathbf{F}$  and determine whether  $\mathbf{F}$  is conservative.

**Solution.** The line integral is  $2\pi$  so  $\mathbf{F}$  is not conservative. However, the curl vanishes. This is not a contradiction because the domain  $U$  where  $\mathbf{F}$  is  $C^1$  is  $\mathbb{R}^2 \setminus \{(0, 0)\}$  which is not simply connected. The vector field  $\mathbf{F}$  is not conservative on this domain. In fact if we consider the simply connected region  $\mathbb{R}^2 \setminus \text{negative real axis}$  we find that  $\mathbf{F}$  is conservative on this region. Indeed,  $\mathbf{F} = \nabla\theta$ .