

**Lecture 3: Review of 20C continued.**

**Lecture 3: 1.4: Cylindrical and Spherical Coordinates.** Recall that in the plane it is sometimes useful to introduce polar coordinates. There are two possible natural and useful generalizations of this to space:

**Cylindrical coordinates**  $(r, \theta, z)$  of a point  $P(x, y, z)$ . Here,  $(r, \theta)$  are just the polar coordinates of  $(x, y)$ . We take  $r \geq 0$  and  $0 \leq \theta < 2\pi$ .

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

To get from  $(r, \theta, z)$  to  $(x, y, z)$  we have

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z.$$

**Geometrically**  $r$  is the distance of  $(x, y, z)$  to the  $z$  axis, and  $\theta$  is the angle between the half of the  $xz$  plane with  $x > 0$  and the half plane containing the  $z$ -axis and the point  $(x, y, z)$  (see picture).

For a constant  $c$ , the surface  $r=c$  is the cylinder radius  $c$  with axis the  $z$ -axis, The surfaces  $\theta=c$  is a half plane from the  $z$ -axis at an angle  $c$  with the  $x$ - $z$  plane. The surface  $z=c$  is a plane parallel to the  $x$ - $y$  plane.

**Spherical coordinates**  $(\rho, \theta, \phi)$  of a point  $P(x, y, z)$ .

**Geometrically**  $\theta$  is just the cylindrical coordinate from before,  $\rho$  is the distance from the point  $(x, y, z)$  to  $(0, 0, 0)$ , so

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

and  $\phi$  is the angle between the positive  $z$ -axis and the position vector  $(x, y, z)$ . Then we see that the cylindrical coordinates are given in terms of the spherical coordinates by

$$r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi.$$

Hence

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

where we must have  $\rho \geq 0$ ,  $0 \leq \theta < 2\pi$ ,  $0 \leq \phi \leq \pi$ . Note that the surfaces  $\rho = c$  are spheres of radius  $c$ . The surfaces  $\theta = c$  were described before. The surfaces  $\phi = c$  are half cones at an angle  $c$  with the positive  $z$ -axis.

**Example 1.** Write the following equations in cylindrical and spherical coordinates.

(a)  $z^2 = x^2 + y^2$ , (b)  $z^2 = x^2 + y^2 + 1$ , (c)  $x^2 + y^2 + z^2 = 4$ .

**Solution.** (a). cylindrical coordinates:  $z^2 = r^2$ . Spherical:  $\phi = \pi/4$  or  $3\pi/4$ .

(b). Cylindrical:  $z^2 = r^2 + 1$ . Spherical:  $\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi + 1$ .

(c)  $z^2 + r^2 = 4$  in cylindrical coordinates and  $\rho^2 = 4$  in spherical coordinates.

**Example 2.** Write down the following inequalities in in cylindrical and spherical coordinates:  $\{(x, y, z); x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$ .

**Solution.** Cylindrical:  $\{(r, \theta, z); 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq z \leq \sqrt{1 - r^2}\}$ .

Spherical coordinates:  $\{(\rho, \theta, \phi); 0 \leq \rho \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2\}$ .

## 2.1 Functions and graphs.

The **graph** of a function  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  is the set of all points  $(x, y, z)$  where  $z = f(x, y)$ .

The **graph** of a function  $f: \mathbf{R}^3 \rightarrow \mathbf{R}$  is the set of all points  $(x, y, z, t)$  where  $t = f(x, y, z)$ .

The **graph** of a function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is the set of all points  $(x_1, x_2, \dots, x_n, t)$  where  $t = f(x_1, x_2, \dots, x_n)$ .

A **level set** of a function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is a set of all points  $(x_1, \dots, x_n)$  such that  $f(x_1, \dots, x_n) = c$ . If  $n = 2$  it is called a level curve  $f(x, y) = c$ . If  $n = 3$  it is called a level surface  $f(x, y, z) = c$ .

**Example.** Draw the level curve  $f(x, y) = x^2 + y^2 = 1$ .

**Solution.** The level curve is a circle.

**Example.** Draw the level surface  $f(x, y, z) = x^2 + y^2 = 1$ .

**Solution.** The level surface is a cylinder since  $z$  is arbitrary.

Note: If you take the level set  $f(x, y) = c$  and lift it up to the plane  $z = c$ , then it becomes the intersection of the graph  $f(x, y) = z$  with the horizontal planes  $z = c$ .

A **section** of a graph of the function  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  is obtained by intersecting the graph  $f(x, y) = z$  with a vertical plane, for example the plane  $x = c$  or  $y = c$ .

Level sets and sections are useful tools to sketch a graph of a function  $z = f(x, y)$ .

**Example.** Sketch the elliptic paraboloid  $z = x^2 + y^2$ .

**Solution.** (1) Sketch the level curves in the plane.

(2) Draw the intersection with the  $y - z$  plane, that is the curve  $x = 0, z = y^2$ .

(3) Draw the level curves  $z = 1, 2, 3, 4$  and raise them to the graph.

**Example.** Sketch the hyperbolic paraboloid (or saddle)  $z = y^2 - x^2$ .

**Solution.** (1) Sketch the level curves in the plane.

(2). Sketch the intersection with the  $y - z$  plane, that is  $z = y^2$ .

Sketch the intersection with the  $x - z$  plane, that is  $z = -x^2$ .

(3). Raise the level curves to the graph.

A **section** of a graph of the function  $f: \mathbf{R}^3 \rightarrow \mathbf{R}$  is obtained by intersecting the graph  $f(x, y, z) = t$  with a plane containing the  $t$ -axis, for example the plane  $x = c$ ,  $y = c$ , or  $z = c$ .

**Example.** Write down the equation of the section of the graph  $f(x, y, z) = x^2 + y^2 - z^2$  in the  $yzt$  hyperplane (that is the plane  $x = 0$ ).

**Solution.**  $t = y^2 - z^2$ .