MATH 20E VECTOR CALCULUS

Lecture 4: Review of 20C continued. 2.2 Limits.

Definition. Suppose that $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$. The equation

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

means that the vector $\mathbf{f}(\mathbf{x})$ approaches $\mathbf{L}$ as the vector $\mathbf{x}$ approaches $\mathbf{a}$. More precisely: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$.

Definition. A function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is called **continuous** at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

Example. At which points is function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$ continuous, where $\mathbf{f}$ is defined by

$$\mathbf{f}(\mathbf{x}) \ = \ \frac{\mathbf{x}}{\|\mathbf{x}\|}?$$

Solution. Write

$$\mathbf{f}(\mathbf{x}) \ = \ \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right)$$

(1). A vector valued function is continuous at $\mathbf{a}$ if each component is continuous at $\mathbf{a}$. We just have to see where the functions $\frac{x}{\sqrt{x^2+y^2}}$ and $\frac{y}{\sqrt{x^2+y^2}}$ are continuous.

(2). The coordinates functions x_1 and x_2 are continuous everywhere.

(3). If functions g and h are continuous at $\mathbf{a}$ then so are $g+h$, $g-h$, gh. In addition so is g/h if $h(\mathbf{a}) \neq 0$. We see in particular that $x^2 + y^2$ is continuous everywhere.

(4). If g is continuous at $\mathbf{a}$ and $h : [a, b] \to \mathbb{R}$ is continuous at $g(\mathbf{a})$ then $h \circ g$ is continuous at $\mathbf{a}$. Hence $\sqrt{x^2 + y^2}$ is continuous everywhere.

(5). We conclude that $\frac{x}{\sqrt{x^2+y^2}}$ and $\frac{y}{\sqrt{x^2+y^2}}$ are continuous at all points with $x^2 + y^2 \neq 0$, that is all points with $(x, y) \neq (0, 0)$.

(6). $\mathbf{f}(\mathbf{x})$ is not continuous at $\mathbf{x} = \mathbf{0}$. Indeed, approaching zero along the x-axis we compute $\mathbf{f}(\varepsilon, 0) = (1, 0)$, while approaching zero along the y-axis we have $\mathbf{f}(0, \varepsilon) = (0, 1)$. Hence there is no unique vector which $\mathbf{f}$ approaches as $\mathbf{x} \to \mathbf{0}$.

2.3. Differentiation.

Given $f : \mathbf{R}^3 \to \mathbf{R}$ The **partial derivative** of f with respect x is defined by

$$f_x(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

if it exist. The partial derivatives $\partial f/\partial y$ and $\partial f/\partial z$ are defined similarly and the extension to functions of n variables is analogous. For $\mathbf{x}_0 = (x_0, y_0, z_0)$ We write $Df(\mathbf{x})$ for the vector

$$Df(\mathbf{x}_0) \;=\; \left[\frac{\partial f}{\partial x}(\mathbf{x}_0), \frac{\partial f}{\partial y}(\mathbf{x}_0), \frac{\partial f}{\partial z}(\mathbf{x}_0)\right].$$

Remark. When we use matrices, we think of $\mathbb{R}^n$ as being a space of column vectors. The vector $\nabla f(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(\mathbf{x}_0) \\ \frac{\partial f}{\partial y}(\mathbf{x}_0) \\ \frac{\partial f}{\partial z}(\mathbf{x}_0) \end{pmatrix}$ in $\mathbb{R}^n$ is called the gradient of f at $\mathbf{x}_0$. We sometimes write it as a row vector so it takes up less space, but when there are matrices around we write it as a column vector.

For the case when $\mathbf{f}$ maps a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, and $\mathbf{a} = (a_1, \ldots, a_n)$, we define

$$\mathbf{Df}(\mathbf{a}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix} = \left[\frac{\partial \mathbf{f}}{\partial x_1}\; \frac{\partial \mathbf{f}}{\partial x_2} \cdots \frac{\partial \mathbf{f}}{\partial x_n}\right]$$

where the partial derivatives are evaluated at $\mathbf{a}$, and we think of $\mathbf{f}(\mathbf{x})$ as a column vector, $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$.

Example Calculate $Df(x, y)$ when $f(x, y) = (x, y, xy)$.
Solution. We have

$$D\mathbf{f}(x, y) \;=\; \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}.$$

Notation

$$f_x = \frac{\partial f}{\partial x}, \qquad \text{etc..}$$

Linear Approximation. **Dimension 1**. $y = f(x)$. Then $f'(x_0) = \dfrac{df}{dx}(x_0)$ is the slope of the graph $y = f(x)$ at the point $(x_0, f(x_0))$. The tangent line to the graph has equation $y = f(x_0) + f'(x_0)(x - x_0)$ and this is the linear function that best approximates $f(x)$ when x is close to x_0. Indeed, f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) \;-\; f(x_0) \;-\; f'(x)(x - x_0)}{\|x - x_0\|} \;=\; \mathbf{0}$$

Dimension 2. $z = f(x, y)$. The partial derivative $\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the graph $z = f(x, y_0)$, where y_0 is fixed.

The **tangent plane** of the graph of f at the point (x_0, y_0) is the plane with equation

$$z \;=\; f(x_0, y_0) \;+\; f_x(x_0, y_0)(x - x_0) \;+\; f_y(x_0, y_0)(y - y_0)$$

A function $f(x, y)$ is called **differentiable** at (x_0, y_0) if the partials exist and if the tangent plane of the graph of f at (x_0, y_0) is a good approximation to the graph itself when (x, y) is close to (x_0, y_0), that is

$$\lim_{(x,y) \to (x_0,y_0)} \frac{f(x, y) \;-\; f(x_0, y_0) \;-\; f_x(x_0, y_0)(x - x_0) \;-\; f_y(x_0, y_0)(y - y_0)}{\|(x, y) - (x_0, y_0)\|} \;=\; \mathbf{0}.$$

Theorem. If the partial derivatives of $f(x, y)$ exists and are continuous for all (x, y) in an open ball centered at (x_0, y_0), then f is differentiable at (x_0, y_0).

Example. Calculate the equation of the tangent plane of the graph of the function $f(x, y) = \sin(x\sqrt{y})$ at the point $(x, y) = (0, 1)$, and use it to estimate $\sin(0.1\sqrt{0.9})$.

Solution. $f(0, 1) = 0$. $f_x = \sqrt{y}\cos(x\sqrt{y})$, so $f_x(0, 1) = 1$. $f_y = x\frac{1}{2}y^{-1/2}\cos(x\sqrt{y})$, so $f_y(0, 1) = 0$. Equation of tangent plane is

$$z \;=\; 0 \;+\; (x - 0) \;+\; 0(y - 1) \;=\; x,$$

so

$$\sin(0.1\sqrt{0.9}) \;\approx\; 0.1.$$

Notice that the equation of the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) can be put into matrix form

$$z \;=\; f(x_0, y_0) + (f_x(x_0, y_0), f_y(x_0, y_0)) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \;=\; f(x_0, y_0) + Df(x_0, y_0)(\mathbf{x} - \mathbf{x}_0).$$

Here,

$$\mathbf{x} \;=\; \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{x}_0 \;=\; \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Definition. A function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is called differentiable at $\mathbf{x}^0$ if

$$\lim_{\mathbf{x} \to \mathbf{x}^0} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0) - \mathbf{Df}(\mathbf{x}^0)(\mathbf{x} - \mathbf{x}^0)}{\|\mathbf{x} - \mathbf{x}^0\|} = 0$$

Here $D\mathbf{f}$ is the matrix defined above, and

$$\mathbf{x} \;=\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{x}^0 \;=\; \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}.$$