Lecture 5: 2.4 Paths and curves.

A **path** in $\mathbb{R}^3$ is a map $\mathbf{c} : I \to \mathbb{R}^3$ of an interval $I = [a, b]$ to $\mathbb{R}^3$, i.e. for each $t \in I$ $\mathbf{c}(t)$ is a vector $\mathbf{c} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. A path is called a **curve** by some authors, but we define a **curve** to be the image in $\mathbb{R}^3$ of a path.

The **velocity** of a path is defined to be

$$\mathbf{c}'(t) = \lim_{h\to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

Since the velocity is tangent to the curve, the **tangent line** to the curve at t_0 is given by

$$\mathbf{L}(t) = \mathbf{c}(t_0) + \mathbf{c}'(t_0)t$$

Example. Let $\mathbf{c}(t) = (\cos t, \sin t, t)$. Find the equation of the tangent line at $t = \pi/2$.

Solution. We have $\mathbf{c}'(t) = (-\sin t, \cos t, 1)$. The equation of the tangent line is

$$\mathbf{L}(t) = \mathbf{c}(\pi/2) + \mathbf{c}'(\pi/2)(t - \pi/2) = (0, 1, \pi/2) + (-1, 0, 1)t$$

or $x = -t$, $y = 1$ and $z = \pi/2 + t$.

2.5. The Chain Rule.

The **Chain rule in one variable**: suppose that $y = g(x)$, and $z = f(y)$, i.e. $z = h(x) := f(g(x)) = f \circ g(x)$ then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} \qquad \Leftrightarrow \qquad h'(x) = f'(y)\, g'(x)$$

Why is this true?

$$\triangle z = f(y + \triangle y) - f(y) \approx f'(y)\triangle y$$

which is the same as saying that f is differentiable. Similarly

$$\triangle y = g(x + \triangle x) - g(x) \approx g'(x)\triangle x$$

and if we combine the two we get

$$\triangle z \approx f'(y)\, g'(x)\triangle x$$

The **Chain rule in several variables**: Suppose that $g : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $\mathbf{x}_0$, and $f : \mathbf{R}^m \to \mathbf{R}^p$ is differentiable at $\mathbf{y}_0 = f(\mathbf{x}_0)$. Then $h = f \circ g : \mathbf{R}^n \to \mathbf{R}^p$ is differentiable at $\mathbf{x}_0$, and

$$Dh(\mathbf{x}_0) = Df(\mathbf{y}_0)\, Dg(\mathbf{x}_0), \qquad \text{where} \qquad \mathbf{y}_0 = g(\mathbf{x}_0).$$

The right hand side is the $p \times n$ matrix equal to the matrix $Df(\mathbf{y}_0)$ multiplied by the matrix $Dg(\mathbf{x}_0)$.

The intuitive argument above actually generalizes to several variables just by replacing f' by Df etc. since differentiability of functions in several variables says

$$g(x + \triangle x) - g(x) \sim Dg(x)\triangle x.$$

Chain rule case 1: $f \circ \mathbf{c}$ where $\mathbf{c}$ is a curve in $\mathbb{R}^3$.
If $h(t) = f(\mathbf{c}(t)) = f(x(t), y(t), z(t))$, then by the chain rule

$$\frac{\partial (f \circ \mathbf{c})}{\partial t} = Df\, D\mathbf{c} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{bmatrix} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

This can also be expressed with the gradient notation and dot product

$$\frac{df \circ \mathbf{c}}{dt}(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$$

Example. If $z = x^2 + y^2$, $x = \cos t$ and $y = \sin t$ find dz/dt.

Solution 1. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 2x\,(-\sin t) + 2y\,\cos t = 2\cos t\,(-\sin t) + 2\sin t \cos t = 0.$

Solution 2. $z = x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so $\frac{dz}{dt} = 0$.

Chain rule case 2:
Example. Consider the surface which is given parametrically by the mapping $(u, v) \to (x, y, z) = (u \cos v, u \sin v, u)$, and the change of variables given by $(p, q) \to (u, v) = (2p, pq)$. Compute the partial derivatives $\partial x/\partial p$, $\partial x/\partial q$, $\partial y/\partial p$, etc., at the point $(p, q) = (1, 0)$ by using the chain rule.

Solution. $Dh = DfDg$,

$$\begin{pmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial q} \\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial p} & \frac{\partial v}{\partial q} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial q} \end{pmatrix} = \begin{pmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ q & p \end{pmatrix}$$

(at $(p, q) = (1, 0)$ we have $(u, v) = (2, 0)$)

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Notice by multiplying out the original matrices we have for example

$$\frac{\partial x}{\partial p} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial p}.$$

$$\begin{tikzcd}
x \arrow[d] \arrow[dr] & y & z \\
u \arrow[d] & v \arrow[dl] & \\
p & q &
\end{tikzcd}$$

To compute $\frac{\partial x}{\partial p}$, using the diagram draw the variables x, y, z on the top line, the variables which they depend on directly (u, v) on the next line and the variables which those variables depend directly on (p, q) on the next line. Draw all the downward paths which join x to p. To each segment of the path, for example the segment between x and v, attach the corresponding partial derivative, in this case $\partial x/\partial v$. Now multiply together the derivatives for each path and sum over all paths.