MATH 20E VECTOR CALCULUS

Lecture 6: 2.6 Gradients and Directional Derivatives.

Review Example. Let $\mathbf{f}(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$, $\mathbf{c}(t) = (1 + t, 3t)$ and $\mathbf{p}(t) = \mathbf{f}(\mathbf{c}(t))$. Find $\mathbf{p}'(t)$.
Solution.

$$\mathbf{p}'(t) = D\mathbf{f}(\mathbf{c}(t))\mathbf{c}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

2.6 The gradient and directional derivative.

The **gradient** of a function $f : \mathbf{R}^3 \to \mathbf{R}$ is the vector

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

However, to save space we usually write it horizonatally

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

unless there are matrices around. (Same as with other vectors in $\mathbb{R}^n$).

Directional Derivatives. Suppose $f(x, y, z)$ is a scalar function on $\mathbb{R}^3$. At a fixed point $\mathbf{x} = (x, y, z)$ we wish to compute the rate at which f changes when we travel in some direction $\mathbf{v}$. Consider the path which passes through $\mathbf{x}$ at time $t = 0$ and travels along a straight line in direction $\mathbf{v}$. This has parametric equation $\mathbf{L}(t) = \mathbf{x} + t\mathbf{v}$. We will differentiate f along this line to compute how it varies.
The **directional derivative** of f at $\mathbf{x}$ along the vector $\mathbf{v}$ is given by

(*) $$\frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\Big|_{t=0}.$$

Applying the chain rule, this is equal to

$$\mathbf{Df}(\mathbf{x})\mathbf{v} \;=\; \frac{\partial f}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial f}{\partial y}\frac{dy}{dt} \;+\; \frac{\partial f}{\partial z}\frac{dz}{dt} \;= \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

When $\mathbf{v}$ is a unit vector, then a particle traveling along the curve $\mathbf{x} + t\mathbf{v}$ travels with unit speed. In this case (*) is called the **directional derivative of f at $\mathbf{x}$ in direction $\mathbf{v}$.** It measures the rate of change of f as you travel from $\mathbf{x}$ in direction $\mathbf{v}$ with unit speed. It is also the slope of the graph obtained by intersecting the graph of f with the vertical plane through $\mathbf{x}$ containing $\mathbf{v}$.

Example 1. The temperature in space is given by $f(x, y, z) = xy^2z^3$. A fly crawls along at unit speed in the direction of the vector $\mathbf{v} = (-1, 1, 0)$ starting at the point

$\mathbf{x} = (2, 1, 1)$. Compute the rate of change of temperature which the fly experiences when it starts off.

Solution. We must compute the directional derivative of f at $\mathbf{x}$ in the direction of the unit vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$. Now $\nabla f = (y^2z^3, 2xyz^3, 3xy^2z^2)$ so $\nabla f(2, 1, 1) = (1, 4, 6)$. Also, $\mathbf{v}/\|\mathbf{v}\| = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Then

$$\frac{df}{ds} = (1, 4, 6) \cdot (\tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0) = \frac{3}{\sqrt{2}}.$$

The gradient points in the direction in which f increases fastest.

In fact the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$, where θ is the angle between ∇f and $\mathbf{v}$, and the max is when $\cos\theta = 1$.

The gradient of f is normal to the level surface of f.

Let $f : \mathbf{R}^3 \to \mathbf{R}$ and let (x_0, y_0, z_0) be a point on the level surface S defined by $f(x, y, z) = k$, for some constant k. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: If $\mathbf{v}$ is a tangent vector to a path $\mathbf{c}(t)$ in S with $\mathbf{c}(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot \mathbf{v} = 0$.
Indeed, since $f(x(t), y(t), z(t)) = k$ it follows that

$$0 = \frac{d}{dt} f(x(t), y(t), z(t)) = \nabla f(x(t), y(t), z(t)) \cdot \mathbf{c}'(t)$$

Let S be a level surface $f(x, y, z) = k$. The **tangent plane** of S at a point (x_0, y_0, z_0) of S is defined by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

In fact (x, y, z) is in the tangent plane if $(x, y, z) - (x_0, y_0, z_0)$ is parallel to the plane and hence perpendicular to the normal.
Example 2. Compute the equation of the tangent plane to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

Solution. Writing $f = x^2 + y^2 - z^2$, we have $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$. Hence $\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. The tangent plane is the plane with this normal passing through $(1, 1, 1)$, which is the plane $2x + 2y - 2z = 2$.

Example 3. What is the maximum possible value of the directional derivative of f at $(2, 1, 1)$ when $f(x, y, z) = xy^2z^3$ and the maximum is over all unit vectors?

Solution. $|\nabla f(2, 1, 1)| = \sqrt{1^2 + 4^2 + 6^2} = \sqrt{53}$.

Example 4. A hill is given by the equation $z = x^2 \sin y$. Compute the (steepest) slope of the hill at the point $(1, 0, 0)$.

Solution. Here, we need to compute the gradient of a function of 2 variables and not a function of 3 variables! The height of the hill z is a function of the two variables x and y. The maximum value of the slope at $(x, y) = (1, 0)$ is the maximum value of dz/ds at $(1, 0)$, which equals $|\nabla z(1, 0)|$. Now $\nabla z = 2x \sin y\mathbf{i} + x^2 \cos y\mathbf{j}$, so $\nabla z(1, 0) = \mathbf{j}$. Hence the maximum slope at $(1, 0)$ is $|\mathbf{j}| = 1$.