MATH 20E VECTOR CALCULUS

Lecture 7: 3.2 Taylor's formula in several variables

Taylors formula for $f : \mathbf{R} \to \mathbf{R}$. If the derivatives $f', f^{(2)}, \dots f^{(k+1)}$ exist and are continuous on the interval from x_0 to $x_0 + h$, then

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + ... + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0, h),$$

where the remainder or error tends to 0 faster than h^k as $h \to 0$. To be precise,

$$R_k(x_0, h) \;=\; \frac{f^{(k+1)}(x')}{(k+1)!}h^{k+1},$$

where x' is between x_0 and $x_0 + h$.

Definition. Writing $x = x_0 + h$, so $h = x - x_0$, the function

$$p_k(x) \;=\; f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + ... + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is called the k^{th} order Taylor approximation to $f(x)$ based at x_0 (or the k^{th} order Taylor polynomial for $f(x)$ based at x_0). It is the polynomial which best approximates f close to x_0. In fact its first k derivatives at x_0 agree with those of f.

Example 1. Calculate the value of $\sin 0.5$ to within 0.001 by approximating the sine function by a Taylor polynomial based at 0.

Solution. First we have to figure out which order Taylor polynomial we have to use in order to get an error of less than 1/1000. To do this, we examine the form of the error given in Taylor's Theorem. Since in our case $f(x) = \sin x$, $x = 0.5$ and $x_0 = 0$, we have $h = x = 0.5$, and

$$\left|\frac{f^{(k+1)}(x')}{(k+1)!}\,h^{k+1}\right| \;=\; \frac{0.5^{k+1}}{(k+1)!}|f^{(k+1)}(x')| \;=\; \frac{1}{2^{k+1}(k+1)!}|f^{(k+1}(x')|.$$

Now we make a table of the values of the derivatives of f, to see how $f^{(k+1)}$ behaves:

$$\begin{aligned}
f(x) &= \sin x,\\
f'(x) &= \cos x,\\
f''(x) &= -\sin x,\\
f^{(3)}(x) &= -\cos x\\
f^{(4)}(x) &= \sin x\\
&\vdots
\end{aligned}$$

We see that $f^{(k+1)}(x')$ is either $\pm\sin x'$ or $\pm\cos x'$, and since these are all bounded by 1, we get that the error is bounded by $1/(2^{k+1}(k+1)!)$. Now we compute

k	$2^{k+1}(k+1)!$
0	2×1
1	4×2
2	8×6
3	16×24
4	$32\times 120 \;>\; 1000.$

We see that the error will be less than $1/1000$ if we approximate $\sin 0.5$ by $f_4(0.5)$. To evaluate this, we compute

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1, \\ f''(0) &= 0, \\ f^{(3)}(0) &= -1 \\ f^{(4)}(0) &= 0. \end{aligned}$$

Then

$$p_4(0.5) \;=\; f(0) + f'(0)\,0.5 + \frac{1}{2}f''(0)\,(0.5)^2 + \frac{1}{6}f^{(3)}(0)\,0.5^3 + \frac{1}{24}f^{(4)}(0)\,0.5^4 \;=\; 0.5 - \frac{0.5^3}{6}.$$

Differentiable functions on $\mathbb{R}^n$. A function $f(x,y,z)$ is continuously differentiable on a set $U\subset\mathbb{R}^3$, if f_x, f_y, f_z exist and are continuous on U. We say that f is twice continuously differentiable on U, if all the second partials $f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yx}, f_{xz}, \ldots$ exist and are continuous on U. In this case, we have $f_{xy} = f_{yx}$, etc..

Taylor polynomials in more variables. We want to approximate the function $f:\mathbb{R}^n\to\mathbb{R}^m$. To make things easy, we will not worry about calculating precise errors. We will just approximate a function f on $\mathbb{R}^n$ by its Taylor polynomials. The first order Taylor polynomial of f based at $(\mathbf{x}^0)$ is the equation of the tangent plane,

$$P_1(\mathbf{x}) \;=\; f(\mathbf{x}^0) + Df(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}^0) \;=\; f(\mathbf{x}^0) + \frac{\partial f}{\partial x_1}(x_1 - x_1^0) + \cdots + \frac{\partial f}{\partial x_n}(x_n - x_n^0).$$

Question: How we find higher order Taylor polynomials, for functions of several variables and bound the error?

Answer. Apply Taylor's formula in one dimension to the function
$g(t) = f(\mathbf{x}_0 + t\mathbf{h})$.

Principle. The kth order Taylor polynomial of f based at $\mathbf{x}_0$ and evaluated at $\mathbf{x}$, which we denote by $P_k(\mathbf{x})$, is equal to the kth order Taylor polynomial of g based at 0 and evaluated at 1, which we denote by $p_k(1)$.

Example 2. Compute the second order Taylor polynomial of $f(x, y)$ based at (x_0, y_0).

Solution. We need to compute

$$p_2(1) \;=\; g(0) \;+\; g'(0) \;+\; \frac{1}{2}g''(0).$$

Now

$$g(t) \;=\; f(x(t), y(t)).$$

Using the chain rule we get

$$g'(t) \;=\; \frac{\partial f}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial f}{\partial y}\frac{dy}{dt} \;=\; f_x\,(x - x_0) \;+\; f_y\,(y - y_0),$$

where the partial derivatives f_x and f_y are evaluated at the point $(x(t), y(t))$. Differentiating again with respect to t using the chain rule, we get

$$\begin{aligned} g''(t) \;&=\; \frac{df_x(x(t), y(t))}{dt}\,(x - x_0) \;+\; \frac{df_y(x(t), y(t))}{dt}\,(y - y_0) \\ &=\; \left(\frac{\partial f_x}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial f_x}{\partial y}\frac{dy}{dt}\right)(x - x_0) \;+\; \left(\frac{\partial f_y}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial f_y}{\partial y}\frac{dy}{dt}\right)(y - y_0) \\ &=\; f_{xx}\,(x - x_0)^2 \;+\; 2f_{xy}(x - x_0)(y - y_0) \;+\; f_{yy}\,(y - y_0), \end{aligned}$$

where again, all the partial derivatives f_{xx}, f_{xy} and f_{yy} are evaluated at $(x(t), y(t))$.

Setting $t = 0$ and evaluating, we get the following formula for $f_2(x, y)$, **where all the partial derivatives are evaluated at the point** (x_0, y_0) (we omit writing out the point each time to save on notation).

$$\begin{aligned} f_2(x, y) \;&=\; g_2(1) \;=\; g(0) \;+\; g'(0) \;+\; \frac{1}{2}g''(0) \\ &=\; f(x_0, y_0) \;+\; f_x(x - x_0) \;+\; f_y(y - y_0) \\ &\qquad +\; \frac{1}{2}f_{xx}(x - x_0)^2 \;+\; f_{xy}(x - x_0)(y - y_0) \;+\; \frac{1}{2}f_{yy}(y - y_0)^2. \end{aligned}$$

Example 3. Let $f(x, y) = 3 + 2x + x^2 + 2xy + 3y^2 + x^3 - y^4$. Find the second degree Taylor polynomial around $\mathbf{x^0} = (0, 0)$.
Solution. The second degree Taylor polynomial is

$$\begin{aligned} &f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\big(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\big) \\ &\qquad = 3 + 2x + \frac{1}{2}\big(2x^2 + 2 \cdot 2xy + 6y^2\big) = 3 + 2x + x^2 + 2xy + 3y^2 \end{aligned}$$

Remark. The same principle works to compute the Taylor polynomials of $f(x, y, z)$ based at (x_0, y_0, z_0). Namely, defining

$$g(t) = f(x_0 + t(x - x_0)\,,\, y_0 + t(y - y_0)\,,\, z_0 + t(z - z_0)),$$

the kth order Taylor polynomial of f based at (x_0, y_0, z_0) and evaluated at (x, y, z), is equal to the kth order Taylor polynomial of g based at 0 and evaluated at 1.