

Lecture 8: 4.2 Arc Length

C^1 curves and regular curves. **Definition.** The path

$$\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is *continuous* at t_0 if

$$\lim_{t \rightarrow t_0} \mathbf{c}(t) = \mathbf{c}(t_0).$$

This is equivalent to the fact that the component functions $x(t), y(t), z(t)$ are all continuous at t_0 . We say $\mathbf{c}(t)$ is *continuous* if it is continuous at each of its points. In this case you can trace it out with a continuous motion of your pen, without making any breaks.

Definition. The path $\mathbf{c}(t)$ is *differentiable* if $\mathbf{c}'(t)$ exists for each t . It is continuously differentiable (C^1) if in addition $\mathbf{c}'(t)$ is continuous.

Recall that the velocity vector $\mathbf{c}'(t)$ is tangent to the image curve at $\mathbf{c}(t)$.

Definition. The differentiable path $\mathbf{c}(t)$ is *regular* at t_0 if $\mathbf{c}'(t) \neq 0$.

Example 1. Determine the regular points of the hypercycloid $x = \cos^3 t, y = \sin^3 t$, where $0 \leq t \leq 2\pi$, and sketch curve.

Solution. $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$, so $\mathbf{c}'(t) = 3 \cos t \sin t (-\cos t, \sin t)$. Hence $\|\mathbf{c}'(t)\| = 3|\cos t \sin t|$ and this is non-vanishing precisely when $t \neq 0, \pi/2, \pi, 3\pi/2, 2\pi$. For the picture, see the book, fig 4.1.2.

Principle. The image curve of the differentiable path \mathbf{c} looks smooth close to $\mathbf{c}(t_0)$ if it is regular at t_0 . Otherwise it can change directions sharply at t_0 .

Remark. The hypercycloid above is called *closed* because $\mathbf{c}(0) = \mathbf{c}(2\pi)$ so it closes up. It is called *simple* because it does not intersect itself.

Arc Length. To compute the arc-length of the curve C , choose a differentiable parameterization of C (if possible), so C is the image of $\mathbf{c}(t)$ for $a \leq t \leq b$. Partition the interval $[a, b]$ by points $a = t_0 < t_1 < \dots < t_n = b$, and approximate the curve by the polygon joining the points $\mathbf{c}(t_0), \mathbf{c}(t_1), \dots, \mathbf{c}(t_n)$. The length of the polygon is

$$\sum_{i=0}^{n-1} |\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)| \approx \sum_{i=0}^{n-1} |\mathbf{c}'(t_i)| \Delta t_i,$$

where $\Delta t_i = t_{i+1} - t_i$. In the limit as we use more and more points, that is $n \rightarrow \infty$, we get

$$L = \int_a^b |\mathbf{c}'(t)| dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} dt$$

Remark. The vector $|\mathbf{c}'(t)|$ is the speed of the path $\mathbf{c}(t)$ at t .

Example 2. Compute the length of the hypercycloid in Example 1.

Solution. $|\mathbf{c}'(t)| = 3|\cos t \sin t|$, so

$$L = 3 \int_0^{2\pi} |\cos t \sin t| dt = 12 \int_0^{\pi} \cos t \sin t dt = 6 \sin^2 t \Big|_0^{\pi} = 6.$$

Remark. Different parameterizations will lead to the same arclength!!

It is often useful to parameterize a curve by arc length measured from some point on the curve. If we already have a path $\mathbf{c}(t)$ parameterizing the curve, then

$$s(t) = \int_{t_0}^t \|\mathbf{c}'(\tau)\| d\tau.$$

If $t(s)$ is the inverse of $s(t)$ then

$$\mathbf{c}_2(s) = \mathbf{c}(t(s))$$

is a reparameterization of the curve in terms of arc length. The new path will move with unit speed. Indeed, by the chain rule $\mathbf{c}'_2(s) = d\mathbf{c}(t(s))/ds = \mathbf{c}'(t(s))dt/ds$. Hence $|\mathbf{c}_2(s)| = |\mathbf{c}'(t)|/(ds/dt) = 1$.

Example 3. Reparameterize the part of the hypercycloid $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ with $0 \leq t \leq \pi/2$ in terms of arclength.

Solution.

$$s(t) = \int_0^t \|\mathbf{c}'(\tau)\| d\tau = 3 \int_0^t \cos \tau \sin \tau d\tau = \frac{3}{2} \sin^2 t.$$

Writing $x(t)$ and $y(t)$ in terms of s we get

$$\mathbf{c}_2(t) = \left(\left(1 - \frac{2s}{3}\right)^{3/2}, \left(\frac{2s}{3}\right)^{3/2} \right).$$

The arc $0 \leq t \leq \pi/2$ corresponds to $0 \leq s \leq 3/2$.