MATH 20E VECTOR CALCULUS

Lecture 9: Section 4.3.

A **Vector field F** is:
- A a vector $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ $= (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ assigned to each point (x,y,z) in space.
- A map $\mathbf{F}: \mathbf{R}^3 \to \mathbf{R}^3$.
- illustrated by drawing the arrow $\mathbf{F}(x,y,z)$ starting at the point (x,y,z) for a few points.

There are many physical examples of vector fields, e.g. the gravitational field at each point in space by Newton's law tells the mass in which direction to accelerate and how much. Similarly, an electric field gives the force on a charged particle. Another example is a sand storm, where the vector field at a point in space $\mathbf{F}(x,y,z)$ gives the velocity of the sand particle at the point (x,y,z). Similarly, the velocity vector field of a fluid gives the velocity of the fluid (particle) at each point in space.

Examples. For a scalar field $f(x,y,z)$, the gradient $\mathbf{F} = \nabla f$ is a vector field, for example if $f = x^2 + y^2 + z^2$, then $\mathbf{F} = (2x, 2y, 2z)$.

Examples. $\mathbf{F}(x,y) = (x,y) = \nabla(x^2+y^2)/2$ is a vector field. On the other hand, $\mathbf{F}(x,y) = (-y,x)$ is a vector field which is not the gradient of a function.

A **Flow line** for a vector field $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$ for all values of t. This means that the $\mathbf{F}$ gives the velocity of the path $\mathbf{c}(t)$ at each point.
Example. Sketch the image curves of the flow lines for the fields $\mathbf{F}(x,y) = (x,y)$ and $\mathbf{F}(x,y) = (-y,x)$.

Solution. The flow lines of the vector field $\mathbf{F}(x,y) = (x,y)$ are straight half-lines which stop at the origin. The flow lines of the vector field $\mathbf{F}(x,y) = (-y,x)$ are circles with center the origin.

The flow lines can not intersect since the vector field gives the direction. Furthermore, through each point there is a unique flow line.
We can often calculate the image curves of flow lines (that is calculate the flow line up to a reparameterization).

Example. Calculate the image curves of the flow lines for vector fields (i). $\mathbf{F}(x,y) = (x,y)$ and (ii). $\mathbf{F}(x,y) = (-y,x)$.

Solution. To calculate some parameterization of the image curve for the flow line of the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, we note that $\mathbf{F}(\mathbf{c}(t))$ should always be parallel to $\mathbf{c}'(t)$, that is

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \lambda(t)\,(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}).$$

The function $\lambda(t)$ is scalar valued and unknown. Equating coefficients and eliminating the scalar $\lambda(t)$, we get the relations

$$\frac{dx/dt}{F_1} = \frac{dy/dt}{F_2} = \frac{dz/dt}{F_3}.$$

(i). For $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ in the plane, we get

$$\frac{dx/dt}{x} \;=\; \frac{dy/dt}{y}.$$

Integrating this along the (unknown) integral curve from $t = t_0$ to $t = t_1$ where $\mathbf{c}(t_0) = (x_0, y_0)$ and $\mathbf{c}(t_1) = (x_1, y_1)$, we get

$$\int_{x_0}^{x_1} \frac{dx/dt}{x}\, dt \;=\; \int_{y_0}^{y_1} \frac{dy/dt}{y}\, dt.$$

Making the substitution $t \to x$ in the first integral and $t \to y$ in the second, we get

$$\int_{x_0}^{x_1} \frac{dx}{x} \;=\; \int_{y_0}^{y_1} \frac{dy}{y}.$$

Carrying out this integral,

$$\ln|x_1| - \ln|x_0| \;=\; \ln|y_1| - \ln|y_0|,$$

so exponentiating up we get

$$\left|\frac{x_1}{x_0}\right| \;=\; \left|\frac{y_1}{y_0}\right|.$$

Now writing $(x_1, y_1) = (x, y)$ for the general point on the image curve of the flow line, we get

$$\left|\frac{x}{x_0}\right| \;=\; \left|\frac{y}{y_0}\right|.$$

(i). For $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ in the plane, we get

$$\frac{dx/dt}{-y} \;=\; \frac{dy/dt}{x}.$$

To get both sides to contain only x or y variables, we write

$$x\frac{dx}{dt} \;=\; -y\frac{dy}{dt}.$$

Integrating this along the (unknown) integral curve from $t = t_0$ to $t = t_1$ where $\mathbf{c}(t_0) = (x_0, y_0)$ and $\mathbf{c}(t_1) = (x_1, y_1)$, we get

$$\int_{x_0}^{x_1} x\frac{dx}{dt}\, dt \;=\; \int_{y_0}^{y_1} -y\frac{dy}{dt}\, dt.$$

Making the substitution $t \to x$ in the first integral and $t \to y$ in the second, we get

$$\int_{x_0}^{x_1} x dx \;=\; \int_{y_0}^{y_1} -y dy.$$

Carrying out this integral,

$$\frac{1}{2}\left(x_1^2 - x_0^2\right) \;=\; -\frac{1}{2}\left(y_1^2 - y_0^2\right),$$

so

$$x_1^2 \;+\; y_1^2 \;=\; x_0^2 \;+\; y_0^2.$$

Now writing $(x_1, y_1) = (x, y)$ for the general point on the image curve of the flow line, we get the equation of a circle

$$x^2 + y^2 \;=\; x_0^2 + y_0^2.$$

Section 4.4. The divergence is defined by

$$\nabla \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial y}. \tag{3}$$

Writing

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} \;+\; \frac{\partial}{\partial y}\mathbf{j} \;+\; \frac{\partial}{\partial z}\mathbf{k},$$

the notation $\nabla \cdot \mathbf{F}$ makes sense.

Example. Calculate the divergence of the following vector fields.
(i). $\mathbf{F}(x, y, z) = (x, y, z)$.
(ii). $\mathbf{F}(x, y) = (x, y)$.
(iii). $\mathbf{F}(x, y) = (-y, x)$.
(iv). $\mathbf{F}(x, y) = \frac{(x,y)}{x^2+y^2}$.
(v). $\mathbf{F}(x, y) = \frac{(x,y)}{(x^2+y^2)^{3/2}}$.

Solution.
(i). 3.
(ii). 2.
(iii). 0.
(iv). 0.
(v). $\frac{-1}{(x^2+y^2)^{5/2}}$.

Note: for (iv) and (v), note that writing $r = \sqrt{x^2 + y^2}$ we have

$$\frac{\partial x}{\partial r} = \frac{x}{r}, \qquad\qquad \frac{\partial y}{\partial r} = \frac{y}{r},$$

so using the chain rule,

$$\nabla \cdot \frac{(x, y)}{r^3} \;=\; \frac{\partial (x/r^3)}{\partial x} \;+\; \frac{\partial (y/r^3)}{\partial y} \;=\; \frac{1}{r^3} \;+\; x\frac{-3}{r^4}\frac{x}{r} \;+\; \frac{1}{r^3} \;+\; y\frac{-3}{r^4}\frac{y}{r} \;=\; \frac{-1}{r^3}.$$

What does the divergence mean?. Let **F** be the velocity field of particles of fluid in the plane. The divergence of the velocity vector field of a fluid is the rate of expansion of the fluid per unit volume. Take a small region R of fluid near (x, y, z) and allow the particles of fluid to flow to new positions. (The particles flow along the flow lines of the vector field **F**.) Then the region R flows to a new region $R(t)$ in time r. Write $V(R(t))$ for the volume of this region.

$$\nabla \cdot \mathbf{F}(x, y, z) \;=\; \lim_{R \to \{(x,y,z)\}} \frac{1}{V(R)} \frac{dV(R(t)}{dt}\bigg|_{t=0}.$$

That is, the divergence of **F** at the point (x, y, z) is given by the limit as the region R shrinks to the point (x, y, z) of the rate of change of the volume of $R(t)$ divided by the Volume of R.
Using this interpretation we can interpret the fact that the divergence in examples (iii), (iv) and (v) is positive, zero and negative respectively.