Math 20E. Midterm 1. April 25, 2004
ANSWER ALL 5 QUESTIONS. Each question is worth 20 points.

1. Let C be the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$, oriented so that when you trace out the triangle, you pass through the points in the order given. Compute the circulation $\int_C \mathbf{F}\cdot d\mathbf{R}$ where $\mathbf{F} = x\mathbf{i} + z\mathbf{j} - y\mathbf{k}$.

Solution 1. The curve consists of 3 line segments:
C_1 joins $(1,0,0)$ to $(0,1,0)$.
C_2 joins $(0,1,0)$ to $(0,0,2)$.
C_3 joins $(0,0,2)$ to $(1,0,0)$.
We can parameterize by
C_1: $(x,y,z) = (1-t,t,0)$
C_2: $(x,y,z) = (0,1-t,2t)$
C_3: $(x,y,z) = (t,0,2-2t)$.
Then

$$\int_C \mathbf{F}\cdot d\mathbf{R} \;=\; \int_C xdx + zdy - ydz.$$

We can save ourselves some work by noticing that $xdx = d(x^2)/2$ is an exact differential and since C is closed, this will not contribute to the answer. Hence

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} \;=\; \int_C zdy - ydz \;&=\; \int_{C_1} zdy - ydz \;+\; \int_{C_2} zdy - ydz \;+\; \int_{C_3} zdy - ydz \\ &= \int_0^1 0dt \;+\; \int_0^1 2t(-1)dt - (1-t)2dt \;+\; \int_0^1 0dt \\ &= \int_0^1 (-2t - 2 + 2t)dt \;=\; \int_0^1 -2\,dt \;=\; -2.\end{aligned}$$

2. For the following vector fields $\mathbf{F}$, either find a scalar field ϕ such that $\mathbf{F} = \nabla\phi$, or show that $\mathbf{F}$ is not conservative on $\mathbb{R}^3$.
(a). $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$.
(b). $\mathbf{F} = yz\mathbf{i} + (xz+z)\mathbf{j} + (xy+y+1)\mathbf{k}$.

Solution 2.
(a). Calculating the curl,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix} \;=\; -y\mathbf{i} \;-\; z\mathbf{j} \;-\; x\mathbf{k}.$$

This is not identically zero, so $\mathbf{F}$ is not conservative.

(b). Solving $\nabla\phi = \mathbf{F}$, we want to solve

$$\frac{\partial\phi}{\partial x} = yz, \tag{1}$$

$$\frac{\partial\phi}{\partial y} = xz + z, \tag{2}$$

$$\frac{\partial\phi}{\partial z} = xy + y + 1 \tag{3}$$

From (1), we have

$$\phi = xyz + C(y,z).$$

Plugging this into (2) gives

$$xz + z = xz + \frac{\partial C}{\partial y},$$

so $\partial C/\partial y = z$ and $C = yz + E(z)$. Hence

$$\phi = xyz + yz + E(z).$$

Plugging this into (3) gives

$$xy + y + 1 = xy + y + \frac{dE}{dz}.$$

Hence $dE/dz = 1$ and $E = z + c$ where c is a constant independent of x, y, z. Putting this together,

$$\phi = xyz + yz + z$$

is a potential function, as one can easily check directly by computing $\nabla\phi$.

3. (a). Calculate the equation of the flow line for the vector field $\mathbf{F} = e^{-x}\mathbf{i} + e^{x}\mathbf{j} + \mathbf{k}$ which passes through the point (x_0, y_0, z_0).

(b). Determine which, if any, of the following three points lie on the same flow line.

$$(0,0,0) \qquad (1,1,1) \qquad \left(1, \tfrac{e^2-1}{2}, e-1\right)$$

Solution 3. (a). We need to solve

$$\frac{dx}{e^{-x}} = \frac{dy}{e^{x}} = dz.$$

From the first equation, we get

$$e^{2x}\,dx = dy.$$

Integrating from (x_0, y_0, z_0) to (x, y, z), we get

$$\frac{1}{2}\left(e^{2x} - e^{2x_0}\right) = y - y_0.$$

Now we consider

$$\frac{dx}{e^{-x}} = dz.$$

Integrating up we get

$$e^x - e^{x_0} = z - z_0.$$

Summarizing, we have

$$\begin{aligned} y &= y_0 + \tfrac{1}{2}\left(e^{2x} - e^{2x_0}\right), \\ z &= z_0 + e^x - e^{x_0}. \end{aligned}$$

(b). The equation of the flow line through $(0, 0, 0)$ is

$$\begin{aligned} y &= \tfrac{1}{2}\left(e^{2x} - 1\right), \\ z &= e^x - 1. \end{aligned}$$

We see by plugging in $x = 1$ that the point $(1, \frac{e^2-1}{2}, e-1)$ lies on the same flow line, but the point $(1, 1, 1)$ does not.

4. For a real valued constant α, let

$$\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\alpha}}.$$

(a). Determine for which values of α the divergence $\nabla \cdot \mathbf{F}$ is positive.

(b). In a region which contains the point $P = (1, 1, 0)$, a fluid has flow

$$\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^2}.$$

No fluid is being created or removed from the system. Is the density of the fluid increasing or decreasing at P?

Solution 4. (a).

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(x(x^2 + y^2 + z^2)^{-\alpha}\right) + \frac{\partial}{\partial y}\left(y(x^2 + y^2 + z^2)^{-\alpha}\right) + \frac{\partial}{\partial z}\left(z(x^2 + y^2 + z^2)^{-\alpha}\right) \\ &= (x^2 + y^2 + z^2)^{-\alpha} - 2\alpha x^2(x^2 + y^2 + z^2)^{-\alpha-1} \\ &\qquad + (x^2 + y^2 + z^2)^{-\alpha} - 2\alpha y^2(x^2 + y^2 + z^2)^{-\alpha-1} \\ &\qquad + (x^2 + y^2 + z^2)^{-\alpha} - 2\alpha z^2(x^2 + y^2 + z^2)^{-\alpha-1} \\ &= 3(x^2 + y^2 + z^2)^{-\alpha} - 2\alpha(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-\alpha-1} \\ &= \frac{3 - 2\alpha}{(x^2 + y^2 + z^2)^{-\alpha}}. \end{aligned}$$

Away from the origin, this is positive exactly when $3 - 2\alpha > 0$, that is $\alpha < 3/2$.

(b). Setting $\alpha = 2$, we see that $\nabla \cdot \mathbf{F}$ is negative away from the origin. This means in particular that it is negative at P, and fluid is tending to flow inwards towards the point P. Since no fluid is being removed at P, this means that the density of the fluid must be increasing at P.

5. (a). Calculate the second order Taylor polynomial of $f(x, y) = y^x = e^{x \ln y}$ based at the point $(x_0, y_0) = (0, 1)$.

(b). Use your answer to (a) to write down an estimate for $0.9^{0.1}$.

Solution 5. (a). Long method

$$f = e^{x \ln y}.$$
$$f_x \;=\; \ln y \; e^{x \ln y}.$$
$$f_{xx} \;=\; (\ln y)^2 \; e^{x \ln y}.$$
$$f_{xy} \;=\; \frac{1}{y} \; e^{x \ln y} \;+\; \ln y \, \frac{x}{y} \, e^{x \ln y}.$$
$$f_y \;=\; \frac{x}{y} \, e^{x \ln y}.$$
$$f_{yy} \;=\; \frac{-x^2}{y} \; e^{x \ln y} \;+\; \left(\frac{x}{y}\right)^2 \; e^{x \ln y},$$

Short cut:

$$f = e^{x \ln y}.$$
$$f_x \;=\; \ln y e^{x \ln y} \;=\; \ln y \; y^x.$$
$$f_{xx} \;=\; (\ln y)^2 y^x.$$
$$f_{xy} \;=\; \frac{1}{y} y^x \;+\; \ln y \, x y^{x-1} \;=\; (1 \;+\; x \ln y) \, y^{x-1}.$$
$$f_y = x y^{x-1}.$$
$$f_{yy} = x(x-1) y^{x-2}.$$

At the point $(0, 1)$,

$$f = 1, \qquad f_x = 0, \qquad f_{xx} = 0, \qquad f_{xy} = 1, \qquad f_y = 0, \qquad f_{yy} = 0.$$

The Taylor polynomial is

$$f_2(x, y) \;=\; 1 \;+\; x(y - 1).$$

(b).

$$f(0.1, 0.9) \;\approx\; f_2(0.1, 0.9) \;=\; 1 \;+\; 0.1(0.9 - 1) \;=\; 0.99.$$