Math 20E. Midterm 2. May 17, 2004
ANSWER ALL 4 QUESTIONS.

1. (25%) Let $\mathbf{F} = e^{x-y}\,\mathbf{i} + (z - e^{x-y})\,\mathbf{j} + y\mathbf{k}$.

(a). Determine whether $\mathbf{F}$ is conservative.

(b). Calculate $\int_C \mathbf{F}\cdot d\mathbf{R}$ where C is the parameterized curve given by

$$x = \tan\left(\frac{\pi t}{4}\right), \qquad y = \ln\left(1 + (e-1)t\right), \qquad z = t^3, \qquad 0 \le t \le 1.$$

(Hint: Use your answer to part (a).)

Solution 1, Method 1. (a). Try to solve $\nabla\phi = \mathbf{F}$, that is

$$\frac{\partial \phi}{\partial x} = e^{x-y}, \tag{1}$$

$$\frac{\partial \phi}{\partial y} = z - e^{x-y}, \tag{2}$$

$$\frac{\partial \phi}{\partial z} = y. \tag{3}$$

Then (1) $\Rightarrow$ $\phi = e^{x-y} + C(y,z)$, but substituting this into (2) we get

$$-e^{x-y} + \frac{\partial C}{\partial y} = z - e^{x-y},$$

and so $\partial C/\partial y = z$, and $C(y,z) = yz + C_1(z)$. Substituting this in to our expression for ϕ,

$$\phi = e^{x-y} + yz + C_1(z).$$

Substituting this into (3), we get

$$y + \frac{dC_1}{dz} = y.$$

Hence C_1 is a constant independent of x, y, z, and a potential for $\mathbf{F}$ is given by

$$\phi = e^{x-y} + yz.$$

It is easy to see that this ϕ works, so $\mathbf{F}$ is conservative.

(b). Since $\mathbf{F} = \nabla\phi$,

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(\text{terminal point}) - \phi(\text{initial point})$$
$$= \phi(x(1), y(1), z(1)) - \phi(x(0), y(0), z(0)) = \phi(1,1,1) - \phi(0,0,0) = 1.$$

Solution 1, Method 2. We note that the vector field $\mathbf{F}$ is continuously differentiable on the whole space. Hence $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$. We compute

$$\nabla \times \mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{x-y} & z - e^{x-y} & y \end{vmatrix} \;=\; \mathbf{0}.$$

Hence $\mathbf{F}$ is conservative.

(b). Because $\mathbf{F}$ is conservative on the whole space,

$$\int_C \mathbf{F} \cdot d\mathbf{R} \;=\; \int_{C'} \mathbf{F} \cdot d\mathbf{R},$$

where C' is any curve with the same endpoints as C. Because the curve C is complicated, we choose a simpler curve with the same endpoints. Now the initial point is $(0,0,0)$ and the terminal point is $(1,1,1)$. Let us choose C' to be the straight line segment $x = t$, $y = t$, $z = t$ with $0 \le t \le 1$. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{R} \;&=\; \int_{C'} e^{x-y}\, dx \;+\; (z - e^{x-y})\, dy \;+\; y\, dz \\ &=\; \int_{t=0}^{1} e^0\, dt + (t - e^0)\, dt + t dt \;=\; \int_0^1 2t\, dt \;=\; 1. \end{aligned}$$

2. (20%) Calculate a vector potential for $\mathbf{F} = (x+y)\mathbf{i} - y\mathbf{j}$.

Solution 2. We have $\mathbf{F} = \nabla \times (\phi \mathbf{k})$, where $\nabla \chi = y\mathbf{i} + (x+y)\mathbf{j}$. This is equivalent to the equations

$$\frac{\partial \chi}{\partial x} = y, \tag{1}$$

$$\frac{\partial \chi}{\partial y} = x + y. \tag{2}$$

From (1) we get $\chi = xy + C(y)$, and plugging this into (2) gives $x + dC/dy = x + y$, so $dC/dy = y$ and $C(y) = y^2/2+$ a constant independent of x, y. Hence $\chi = xy + y^2/2$ is a solution, and $(xy + y^2/2)\mathbf{k}$ is a vector potential for $\mathbf{F}$.

3. (25%) By substituting $u = xy$ and $v = y/x$, calculate the area of the region S given by

$$1 \le xy \le 2, \qquad \frac{1}{2} \le \frac{y}{x} \le 2, \qquad (x > 0,\; y > 0).$$

(You may assume that the mapping $(x,y) \to (u,v)$ is one-to-one from the quadrant $x > 0$, $y > 0$ onto the quadrant $u > 0$, $v > 0$.)

Solution 3.

$$\frac{\partial(u,v)}{\partial(x,y)} \;=\; \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \;=\; \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} \;=\; \frac{2y}{x}.$$

Hence

$$dxdy \;=\; \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \;=\; \frac{x}{2y}\,dudv.$$

Then the area of S is

$$\iint_S 1\,dxdy \;=\; \int_{u=1}^{2}\int_{v=1/2}^{2} \frac{x}{2y}\,dvdu.$$

We must express x/y in terms of u and v, but it is just $1/v$. Hence

$$\iint_S 1\,dxdy \;=\; \int_{u=1}^{2}\int_{v=1/2}^{2} \frac{1}{2v}\,dvdu \;=\; \frac{(\ln 2 - \ln 1/2)}{2} \;=\; \ln 2.$$

4. (30%) Let $\mathbf{F} = \mathbf{j}$. For the following choices of S and $\mathbf{n}$, calculate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

(a). S is the piece of the plane $z = y+2$ with $x^2+y^2 \le 4$, and $\mathbf{n}$ is the unit normal with $\mathbf{n}\cdot\mathbf{k} > 0$.

(b). S is the piece of cylinder $x^2+y^2 = 4$ with $-2 \le z \le y+2$, and $\mathbf{n}$ is the outward unit normal.

Solution 4. (a). S is a graph above the disc Ω in the x–y plane given by $x^2+y^2 \le 4$. To compute the unit normal $\mathbf{n}$, we write S as $z - y = 2$ and take the gradient to get normal $-\mathbf{j}+\mathbf{k}$. Hence

$$\mathbf{n} \;=\; \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}}.$$

Then

$$\mathbf{F}\cdot\mathbf{n} \;=\; \mathbf{j}\cdot\mathbf{n} \;=\; \frac{-1}{\sqrt{2}},$$

and

$$dS \;=\; \frac{1}{|\mathbf{n}\cdot\mathbf{k}|}\,dxdy \;=\; \sqrt{2}\,dxdy.$$

Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \;=\; \iint_\Omega -1\,dxdy \;=\; -\text{Area of }\Omega.$$

Since Ω is a disc of radius 2, the answer is -4π.

(b). For the cylinder $x^2+y^2 = 4$, we have

$$dS \;=\; 2\,d\theta dz.$$

Moreover,

$$\mathbf{n} \;=\; \frac{1}{2}(x\mathbf{i} \;+\; y\mathbf{j}).$$

Hence

$$\mathbf{F}\cdot\mathbf{n} \;=\; \mathbf{j}\cdot\mathbf{n} \;=\; \frac{y}{2}.$$

Then

$$\iint_S \mathbf{F}\cdot d\mathbf{R} \;=\; \int_{\theta=0}^{2\pi}\int_{z=-2}^{y+2} y\,dz d\theta.$$

Writing y in terms of the parameters θ and z, we get $y = 2\sin\theta$, and

$$\int_{\theta=0}^{2\pi}\int_{z=-2}^{2\sin\theta+2} 2\sin\theta\,dz d\theta \;=\; \int_{\theta=0}^{2\pi} 2\sin\theta(2\sin\theta+4)\,d\theta \;=\; 4\pi.$$