MATH 20E MIDTERM 2 SOLUTIONS
Winter 1996

1. Let $\phi = x^2 - z$ and let $\psi = \dfrac{1}{x} + z$.
Find a direction $\mathbf{B}$ so that ϕ and ψ both *increase* when you move away from the point $(1,1,1)$ in the direction $\mathbf{B}$.

SOLUTION: Saying that ϕ is increasing in direction $\mathbf{B}$ is the same as saying that the directional derivative of ϕ in the direction $\mathbf{B}$ is positive, *i.e.* $\dfrac{d\phi}{ds}_{(\mathbf{B})}(1,1,1) > 0$. Strictly speaking, the directional derivative is only defined for unit vectors, so we should say instead $\dfrac{d\phi}{ds}_{(\mathbf{B}/|\mathbf{B}|)}(1,1,1) > 0$.

Now we have the formula

$$\frac{d\phi}{ds}_{(\mathbf{u})} = \nabla\phi \cdot \mathbf{u}.$$

So ϕ increases in direction $\mathbf{B}$ is expressed by:

$$\nabla\phi(1,1,1) \cdot \frac{\mathbf{B}}{|\mathbf{B}|} > 0,$$

which is equivalent to

$$\nabla\phi(1,1,1) \cdot \mathbf{B} > 0.$$

If we want both ϕ *and* ψ to increase in the direction $\mathbf{B}$, we get the conditions

$$\nabla\phi(1,1,1) \cdot \mathbf{B} > 0 \qquad \text{and} \qquad \nabla\psi(1,1,1) \cdot \mathbf{B} > 0.$$

We have $\nabla\phi = 2x\mathbf{i} - \mathbf{k}$, so $\nabla\phi(1,1,1) = 2\mathbf{i} - \mathbf{k}$, and $\nabla\psi = (-1/x^2)\mathbf{i} + \mathbf{k}$, so $\nabla\psi(1,1,1) = -\mathbf{i} + \mathbf{k}$.

Let's write $\mathbf{B} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, so our condition becomes

$$(2\mathbf{i} - \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) > 0 \qquad \text{and} \qquad (-\mathbf{i} + \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) > 0.$$

After doing the dot product, this gives $2a - c > 0$ and $-a + c > 0$, and putting this together we get $a < c < 2a$.

We can choose any vector $\mathbf{B}$ satisfying this condition, for example, $a = 2$, $c = 3$ so $\mathbf{B} = 2\mathbf{i} + 3\mathbf{k}$ or $a = 2.5$, $b = -1$, $c = 3$, so $\mathbf{B} = 2.5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

2. Consider the vector field $\mathbf{F} = \dfrac{y\mathbf{i}}{x^2 + y^2} - \dfrac{x\mathbf{j}}{x^2 + y^2}$ in the plane.

a). Calculate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

SOLUTION:

$$\nabla \cdot \left(\frac{y\mathbf{i}}{x^2+y^2} - \frac{x\mathbf{j}}{x^2+y^2} \right) = \frac{\partial}{\partial x}\frac{y}{x^2+y^2} - \frac{\partial}{\partial y}\frac{x}{x^2+y^2}$$
$$= \frac{\partial}{\partial x} y(x^2+y^2)^{-1} - \frac{\partial}{\partial y} x(x^2+y^2)^{-1}$$
$$= -y(x^2+y^2)^{-2}(2x) + x(x^2+y^2)^{-2}(2y)$$
$$= -2xy(x^2+y^2)^{-2} + 2xy(x^2+y^2)^{-2} = 0.$$

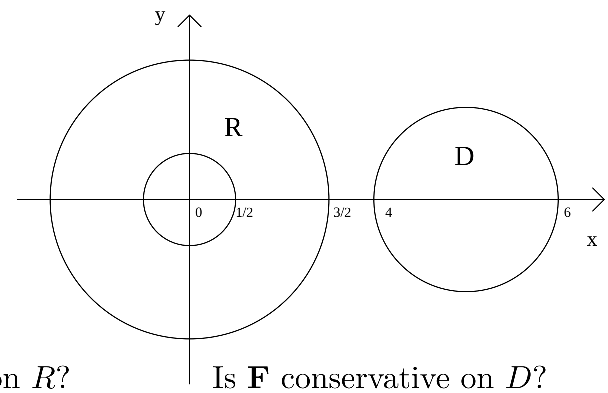
$$\nabla \times \left(\frac{y\mathbf{i}}{x^2+y^2} - \frac{x\mathbf{j}}{x^2+y^2} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 \end{vmatrix}$$
$$= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial}{\partial x}(-\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \right)\mathbf{k}$$
$$= -\left(\frac{\partial}{\partial x} x(x^2+y^2)^{-1} + \frac{\partial}{\partial y} y(x^2+y^2)^{-1} \right)\mathbf{k}$$
$$= -\left((x^2+y^2)^{-1} - x(x^2+y^2)^{-2}(2x) + (x^2+y^2)^{-1} - y(x^2+y^2)^{-2}(2y) \right)\mathbf{k}$$
$$= -\left(2(x^2+y^2)^{-1} - 2(x^2+y^2)(x^2+y^2)^{-2} \right)\mathbf{k} = -\left(2(x^2+y^2)^{-1} - 2(x^2+y^2)^{-1} \right)\mathbf{k} = \mathbf{0}.$$

b). Calculate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the circle with center $(0,0)$ and radius 1 directed anticlockwise.

SOLUTION: First parameterize C. (Anticlockwise means counter-clockwise.) $x = \cos\theta$, $y = \sin\theta$, $0 \leq \theta \leq 2\pi$. Now we write everying in terms of θ.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \frac{y}{x^2+y^2}\, dx - \frac{x}{x^2+y^2}\, dy$$
$$= \int_0^{2\pi} \frac{\sin\theta}{1}\frac{d\cos\theta}{d\theta}\, d\theta - \frac{\cos\theta}{1}\frac{d\sin\theta}{d\theta}\, d\theta$$
$$= \int_0^{2\pi} -\sin^2\theta - \cos^2\theta\, d\theta$$
$$= \int_0^{2\pi} -1\, d\theta = -2\pi.$$

Consider the annulus R and a disc D shown below.



c). Is $\mathbf{F}$ conservative on R? Is $\mathbf{F}$ conservative on D?

SOLUTION: $\mathbf{F}$ is not conservative on R, since in part b) we found that $\int_C \mathbf{F} \cdot \mathbf{R} \neq 0$ for the closed curve C in R.
$\mathbf{F}$ is conservative on D, since $\nabla \times \mathbf{F} = \mathbf{0}$ on D, and D is simply connected.

3.

$$\mathbf{F} = \frac{x}{x^2+z^2}\mathbf{i} + y\mathbf{j} + \frac{z}{x^2+z^2}\mathbf{k}.$$

a). Let C be the flow line of $\mathbf{F}$ through the point $(2, e, 2)$. Parameterize C with x as the parameter.

SOLUTION: The flow line through $(2, e, 2)$ is the curve through $(2, e, 2)$ whose tangent is always parallel to $\mathbf{F}$, *i.e.*
$dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ is parallel to $\frac{x}{x^2+z^2}\mathbf{i} + y\mathbf{j} + \frac{z}{x^2+z^2}\mathbf{k}$. This can be written as

$$\frac{dx}{\frac{x}{x^2+z^2}} = \frac{dy}{y} = \frac{dz}{\frac{x}{x^2+z^2}}.$$

To solve these equations, we first choose one which involves only 2 variables:

$$\frac{dx}{\frac{x}{x^2+z^2}} = \frac{dz}{\frac{x}{x^2+z^2}} \qquad \Rightarrow \qquad \frac{(x^2+z^2)dx}{x} = \frac{(x^2+z^2)dz}{z}.$$

We then reduce this equation so that each side only involves one of the variables. In this case, we divide by $x^2 + z^2$ to get $\frac{dx}{x} = \frac{dz}{z}$. We can now integrate, and get $\ln|x| + C = \ln|z|$. Plugging in the point $(2, e, 2)$, we get $\ln|2| + C = \ln|2|$, so $C = 0$ and $\ln|x| = \ln|z|$. Exponentiating we get $|x| = |z|$, and so $z = \pm x$. Plugging in $(2, e, 2)$ again, we get

$$z = x.$$

We now plug *this* into the equation $\frac{dx}{\frac{x}{x^2+z^2}} = \frac{dy}{y}$, and get

$$\frac{dx}{\frac{x}{x^2+x^2}} = \frac{dy}{y} \qquad \Rightarrow \qquad 2xdx = \frac{dy}{y}.$$

Integrating, we get $x^2+C = \ln|y|$. Plugging in $(2, e, 2)$ we get $4+C = 1$, so $C = -3$ and $x^2 - 3 = \ln|y|$. Exponentiating we get

$$y = e^{x^2-3}.$$

b). Find a scalar field ϕ with $\nabla\phi = \mathbf{F}$ and $\phi(2, e, 2) = 0$.

SOLUTION: We will solve the equations

$$(1) \qquad \frac{\partial\phi}{\partial x} = \frac{x}{x^2+z^2}$$

$$(2) \qquad \frac{\partial\phi}{\partial y} = y$$

$$(3) \qquad \frac{\partial\phi}{\partial z} = \frac{z}{x^2+z^2}.$$

From (1), we get

$$\phi = \int \frac{x}{x^2+z^2}\, dx,$$

where z is held constant in the integration. Substituting $u = x^2 + z^2$, we have $du = 2xdx$ and

$$(4) \qquad \phi = \int \frac{du}{2u} = \frac{1}{2}\ln|u| + C(y, z) = \frac{1}{2}\ln|x^2 + z^2| + C(y, z).$$

The constant $C(y, z)$ might depend on y and z, since y and z are held as constants when we partially differentiate with respect to x.

From this expression (4) for ϕ, we calculate

$$\frac{\partial\phi}{\partial y} = \frac{\partial\frac{1}{2}\ln|x^2+z^2| + C(y,z)}{\partial y} = \frac{\partial C(y,z)}{\partial y}.$$

But we want to satisfy (2):

$$\frac{\partial\phi}{\partial y} = y$$

so we get

$$\frac{\partial C(y,z)}{\partial y} = y,$$

and integrating this we get $C(y,z) = \frac{1}{2}y^2 + C_0(z)$. The constant C_0 might depend on z (it can't depend on x because $C(y,z)$ didn't!). Putting this expression for $C(y,z)$ in (4) we get

$$\phi \;=\; \frac{1}{2}\ln|x^2+z^2| \;+\; \frac{1}{2}y^2 \;+\; C_0(z).$$

From this expression for ϕ, we calculate

$$\frac{\partial\phi}{\partial z} \;=\; \frac{\partial\frac{1}{2}\ln|x^2+z^2|}{\partial z} \;+\; \frac{dC_0(z)}{dz} \;=\; \frac{z}{x^2+z^2} \;+\; \frac{dC_0(z)}{dz}.$$

Comparing this with the equation (3) we want to satisfy, we see that

$$\frac{\partial\phi}{\partial z} \;=\; \frac{z}{x^2+z^2}$$

we get $dC_0(z)/dz = 0$, and $C_0(z) = C_1$ is a constant independent of z. Finally then,

$$\phi \;=\; \frac{1}{2}\ln|x^2+z^2| \;+\; \frac{1}{2}y^2 \;+\; C_1.$$

Now we satisfy the condition $\phi(2,e,2) = 0$. Plugging in $(2,e,2)$, we have

$$0 \;=\; \frac{1}{2}\ln|2^2+2^2| \;+\; \frac{1}{2}e^2 \;+\; C_1,$$

so $C_1 = -\frac{1}{2}\ln 8 - \frac{1}{2}e^2$.

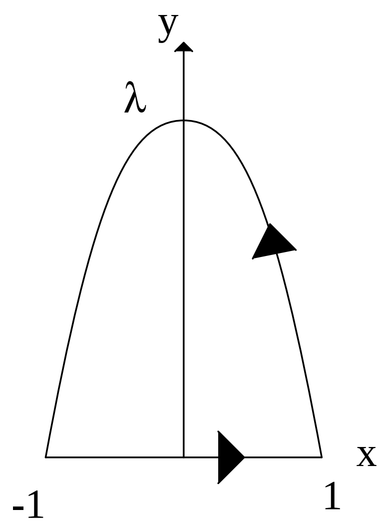
c). What is the angle between the surface $\phi = 0$ and the curve C from part a), at the point $(2,e,2)$?

SOLUTION: Since C is a flow line for $\mathbf{F}$, at $(2,e,2)$, C has tangent $\mathbf{F}(2,e,2)$. But $\nabla\phi = \mathbf{F}$, so this is $\nabla\phi(2,e,2)$. But $\nabla\phi(2,e,2)$ is perpendicular to the surface $\phi = 0$, so C is perpendicular to the surface $\phi = 0$ at $(2,e,2)$.

4. C is the closed curve which consists of the interval from $(-1,0)$ to $(1,0)$ together with the arc of the parabola

$$y = \lambda(1-x^2), \qquad -1 \le x \le 1,$$

traversed anticlockwise. Here, λ is just a constant. D is the region bounded by the curve C.



$\mathbf{F} = x^3\mathbf{j}$.

a). Calculate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R}$.

SOLUTION: The curve C is made up of a straight part C_1 and a curved part C_2.

$$\int_C \mathbf{F} \cdot d\mathbf{R} \; = \; \int_C x^3\, dy \; = \; \int_{C_1} x^3\, dy \; + \; \int_{C_2} x^3\, dy.$$

Now on C_1, the variable y does not change, so $dy = 0$ and $\int_{C_1} x^3\, dy = 0$. We will parameterize C_2. We can use x as the parameter, then $y = \lambda(1 - x^2)$. We have to be careful, since the initial point is $x = 1$ and the final point is $x = -1$. We get

$$\int_{C_2} x^3\, dy \; = \; \int_1^{-1} x^3 \frac{d\lambda(1 - x^2)}{dx}\, dx \; = \; \int_1^{-1} x^3(-2\lambda x)\, dx$$

$$= \; -2\lambda \int_1^{-1} x^4\, dx \; = \; -2\lambda \frac{x^5}{5}\bigg|_{x=1}^{x=-1} \; = \; \frac{-2\lambda}{5}((-1)^5 - 1^5) \; = \; \frac{4\lambda}{5}.$$

b). Calculate $\nabla \times \mathbf{F}$.

SOLUTION:

$$\nabla \times x^3\mathbf{j} \; = \; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^3 & 0 \end{vmatrix} = \; 0\mathbf{i} \; + \; 0\mathbf{j} \; + \; \frac{\partial x^3}{\partial x}\mathbf{k} = 3x^2\mathbf{k}.$$

c). Without using your answer to a), calculate $\displaystyle\int_D \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$.

SOLUTION: The region D can be described by $-1 \leq x \leq 1$, $0 \leq y \leq \lambda(1-x^2)$. Hence

$$\int_D \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA \;=\; \int_D 3x^2\, dA \;=\; \int_{x=-1}^{1} \int_{y=0}^{\lambda(1-x^2)} 3x^2\, dy dx \;=\; \int_{x=-1}^{1} 3x^2 y \Big|_{y=0}^{\lambda(1-x^2)} dx$$

$$= \int_{x=-1}^{1} 3x^2 \lambda(1-x^2)\, dx \;=\; 3\lambda \int_{x=-1}^{1} x^2 - x^4\, dx \;=\; 3\lambda(\frac{x^3}{3} - \frac{x^5}{5})\Big|_{-1}^{1}$$

$$= 3\lambda(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}) \;=\; \frac{4\lambda}{5}.$$