Solutions to Math 20E Midterm 1, Winter 2003, Lindblad.

1. (b) The flow line starts going in the direction of $\mathbf{F}(1,0) = \mathbf{i} + \mathbf{j}$. We can either graphically solve for the flow lines by plotting the vector field at more points and sketching the curve from the starting point that everywhere goes in the direction of the vector field, or we can check to see if the flow line is equal to any of the given curves by finding the tangent line of these. Calculating the tangent line for the 4 curves when $t = 0$ we see that only (2) and (4) starts in the right direction. (4) is however a line and since the vector field changes and turns in the mathematically negative direction along the line we must exclude (4). Therefore it can only be (2). By solving it graphically or sketching the curve (2) we see that the flow line is a spiral going outward in the mathematically positive direction for positive times.

2. (a). $\nabla \cdot \mathbf{F} = a + d$. (b). $\nabla \times \mathbf{F} = (c - b)\mathbf{k}$.
(c). $\mathbf{F}$ is conservative when $\nabla \times \mathbf{F} = 0$, i.e. when $c - b = 0$.
We must find ϕ such that $\phi_x = ax + by$ and $\phi_y = cx + dy$. Integrating the first equation gives $\phi = ax^2/2 + bxy + f(y)$ for any function $f(y)$. It follows that $\phi_y = bx + f'(y)$ and using the second equation we see that we must have $f'(y) = dy$. It follows that $\phi = ax^2/2 + bxy + dy^2/2$.

3.
$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} \Big((ax + by)\frac{dx}{dt} + (cx + dy)\frac{dy}{dt} \Big)\, dt$$
$$= \int_0^{2\pi} \Big((ar\cos t + br\sin t)(-r\sin t) + (cr\cos t + dr\sin t)r\cos t \Big)\, dt$$
$$= \int_0^{2\pi} r^2 \Big((d - a)\cos t \sin t + c\cos^2 t - b\sin^2 t \Big)\, dt$$
$$= \int_0^{2\pi} r^2 \Big((d - a)\sin 2t + c\frac{1 + \cos 2t}{2} - b\frac{1 - \cos 2t}{2} \Big)\, dt$$
$$= \int_0^{2\pi} r^2 \Big((d - a)\cos t \sin t + c\cos^2 t - b\sin^2 t \Big)\, dt$$
$$= r^2 \Big(-(d - a)\frac{\cos 2t}{2} + c\frac{2t + \sin 2t}{4} - b\frac{2t + \sin 2t}{4} \Big)\Big|_0^{2\pi} = \pi r^2 (c - b)$$

4. (a) $\mathbf{F} = \nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} = 2x\mathbf{i} + 4y\mathbf{j}$.
(b) We must find the flow lines for the vector field $\mathbf{F} = 2x\mathbf{i} + 4y\mathbf{j}$: $dx/2x = dy/4y$ which is equivalent to $\ln|x|/2 = \ln|y|/4 + C_1$ or $\ln|y| = \ln|x|^2 - 4C_1$. Exponentiating both sides gives $|y| = |x|^2 e^{-4C_1}$ or for some other constant C: $y = Cx^2$. Since we want to find the flow line that passes through $(x, y) = (1, 2)$ we must have $2 = C$ so the flow line is $y = 2x^2$.