20E Midterm 1 Solutions... and incorrect solutions.

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Q1. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a function such that $(\partial\phi/\partial x)(0,0,0) = 2$, $(\partial\phi/\partial y)(0,0,0) = 3$ and $(\partial\phi/\partial z)(0,0,0) = 4$.

a) Let $w(t) := \phi[c(t)]$, where $c(t) := (t, t^2, 3t)$. Find $(dw/dt)(0)$.

b) In which direction is the greatest rate of increase of ϕ at the point $(0,0,0)$?

c) Let $F = \nabla\phi$. Find $\nabla \times F$.

Solution:

a) By the chain-rule,

$$(dw/dt)(t) = (\nabla(\phi))(c(t)) \cdot c'(t)$$

$$(dw/dt)(0) = (\nabla(\phi))(c(0)).c'(0).$$

Now, $c(t) := (t, t^2, 3t)$. Thus $c'(t) = (1, 2t, 3)$ and $c(0) = (0,0,0)$. $c'(0) = (1, 0, 3)$. Thus

$$\begin{aligned}(dw/dt)(0) &= (\nabla(\phi))(0,0,0).(1,0,3)\\ &= (2,3,4).(1,0,3)\\ &= 2+12 = 14.\end{aligned}$$

Note that the answer must be a scalar, not a vector. There must therefore be no $\hat{i}$'s, $\hat{j}$'s or $\hat{k}$'s in this answer. Also (dw/dt) is evaluated at $t=0$, so the answer must not contain any t's.

b) The greatest rate of increase is in the direction $(\nabla(\phi))(x,y,z)$, at the point (x,y,z). Thus the greatest rate of increase at $(0,0,0)$ is $(\nabla(\phi))(0,0,0) = (2,3,4)$. Note that $(2,3,4) = 2\hat{i} + 3\hat{j} + 4\hat{k}$, and that $(2\hat{i}, 3\hat{j}, 4\hat{k})$ makes no sense.

c) Theorem: Let ϕ have two continuous derivatives. Then $\nabla \times (\nabla(\phi)) = 0$.

Proof:

$$(\nabla(\phi))(x,y,z) = (\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z)$$

Thus

$$\nabla \times (\nabla(\phi)) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right]$$
$$- \hat{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right]$$
$$+ \hat{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right]$$

Since ϕ is twice differentiable, $\partial^2\phi/\partial x\partial z = \partial^2 \phi/\partial z \partial x$, etc. Thus

$$= 0.$$

In your solution you did not need to prove the theorem, just state it. The following is the most common solution, that I saw. It is also incorrect.

Incorrect solution:

$\nabla(\phi) = (2,3,4)$, therefore $\nabla \times \nabla(\phi) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2 & 3 & 4 \end{vmatrix} = 0$. This is not correct since we do not actually know what $\nabla(\phi)$ is - except at the point $(0,0,0)$, where $(\nabla(\phi))(0,0,0) = (2,3,4)$.

Question 2: Let $f(x,y) := x\cos(x+y)$

a) Calculate the second order Taylor polynomial of f about $(1,-1)$.

b) Use a) to estimate $f(1.1,-0.8)$

c) Use a linear approximation of f to approximate $f(1.1,-0.8)$.

Solution:

a)

$$f(x,y) = x\cos(x+y) \Rightarrow f(1,-1) = 1$$

$$(\partial f/\partial x)(x,y) = \cos(x+y) - x\sin(x+y)$$

$$\Rightarrow (\partial f/\partial x)(1,-1) = 1$$

$$(\partial f/\partial y)(x,y) = -x\sin(x+y)$$

$$\Rightarrow (\partial f/\partial y)(1,-1) = 0$$

$$(\partial^2 f/\partial x^2)(x,y) = -\sin(x+y) - \sin(x+y) - x\cos(x+y)$$

$$\Rightarrow (\partial^2 f/\partial x^2)(1,-1) = -1$$

$$(\partial^2 f/\partial x\partial y)(x,y) = -\sin(x+y) - x\cos(x+y)$$

$$\Rightarrow (\partial^2 f/\partial x\partial y)(1,-1) = -1$$

$$(\partial^2 f/\partial y^2)(x,y) = -x\cos(x+y)$$

$$\Rightarrow (\partial^2 f/\partial y^2)(1,-1) = -1.$$

The second order Taylor polynomial of f at (x^0, y^0) is given by

$(T_2(f))(x,y)$

$$= f(x^0, y^0) + (\partial f/\partial x)(x^0, y^0)(x - x^0)$$
$$+ (\partial f/\partial y)(x^0, y^0)(y - y^0)$$
$$+ \frac{(\partial^2 f/\partial x^2)(x^0, y^0)(x - x^0)^2}{2}$$
$$+ \frac{(\partial^2 f/\partial y^2)(x^0, y^0)(x - y^0)^2}{2}$$
$$+ (\partial^2 f/\partial x \partial y)(x^0, y^0)(x - x^0)(y - y^0).$$

Thus when $(x^0, y^0) = (1, -1)$

$$= 1 + 1 . (x - x^0) + 0 . (y - y^0)$$
$$+ (-1)\frac{(x - x^0)^2}{2} + (-1)\frac{(y - y^0)^2}{2}$$

$$+ (-1)(x - x^0)(y - y^0)$$

$$= 1 + (x - 1) + \frac{(-1)(x-1)^2}{2}$$

$$+ \frac{(-1)(y+1)^2}{2} + (-1)(x-1)(y+1)$$

b) We want

$$(T_2(f))(1.1, -0.8)$$

$$= 1 + (0.1) + \frac{(-1)(0.1)^2}{2}$$

$$+ \frac{(-1)\cdot(0.2)^2}{2} + (-1)(0.1)(0.2)$$

$$= ?$$

c) Use only the linear terms... that is, the ones without squares, and also we do not want the $(x-1)(y+1)$ term

The answer, then, is

$$= 1 + (0 \cdot 1)$$

$$= 1.1.$$

Question 3: Let $G(x,y) := (-y, x)$

a) Show that $c(t) := (r\cos(t), r\sin(t))$ is a flow line for G.

b) Sketch the vector-field at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$; also sketch the flow line through $(1,0)$.

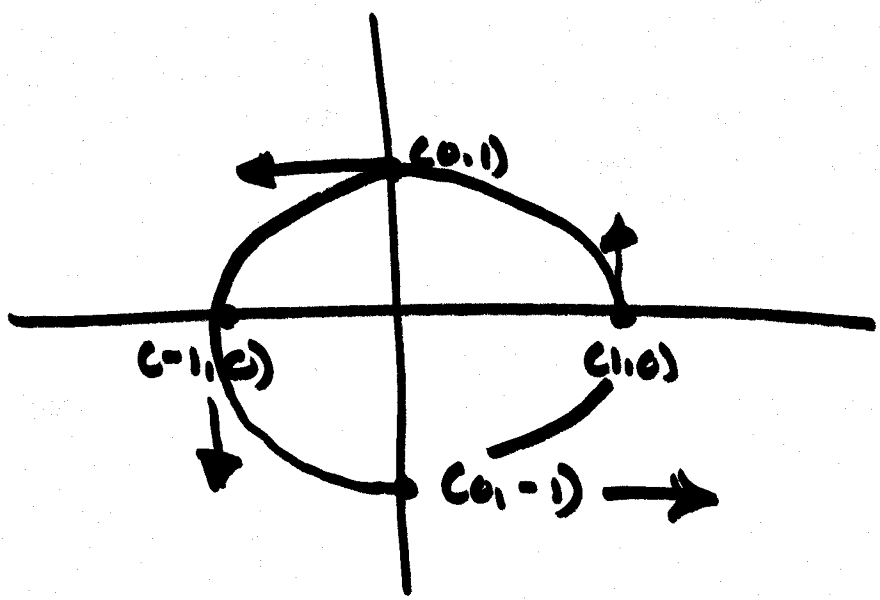
Solution: we know that c is a flow line if $G(c(t)) = (dc/dt)(t)$.

We have $G(c(t)) = G(r\cos(t), r\sin(t))$

$$= (-r\sin(t), r\cos(t)).$$

Also, $(dc/dt)(t) = (-r\sin(t), r\cos(t))$.
Thus c is a flow line.

b)



A common solution for part a(which was worth nothing. or very little):

$$G(c(t)) = c'(t)$$

$$(-r\sin(t), r\cos(t)) = (-r\sin(t), r\cos(t))$$

This is not a solution because the first line is not always true. If something is not true you must state why you are writing it down. the second line is true but trivial. Explain what you are up to...

Question 4: Define $F(x,y,z) := ((y^2+x), -(x^2-y), z)$.

a) Compute $\nabla \times F$

b) Compute $\nabla \cdot F$

c) Compute $D(F)$.

a) Curl is a vector, so make sure your answer is a vector.

$$(\nabla \times F)(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+x & -x^2+y & z \end{vmatrix}$$

$$= \hat{i}[0] - \hat{j}[0] + \hat{k}[-2x - 2y]$$

b) $\nabla \cdot F$ is a scalar. Thus your answer must not contain any $\hat{i}$'s, $\hat{j}$'s or $\hat{k}$'s.

$$(\nabla \cdot F)(x,y,z) = \partial/\partial x\,(y^2+x) + \partial/\partial y\,(-x^2+y) + \partial/\partial z\,(z) = 3.$$

c)

$$(D(F))(x,y) = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y & \partial F_1/\partial z \\ \partial F_2/\partial x & \partial F_2/\partial y & \partial F_2/\partial z \\ \partial F_3/\partial x & \partial F_3/\partial y & \partial F_3/\partial z \end{pmatrix},$$

where $F = (F_1, F_2, F_3)$

$$= \begin{pmatrix} 1 & 2y & 0 \\ -2x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$