Solutions for Math 20E Midterm 2, Fall 98, Lindblad.

1. (a) The equation for S is $x+y+z=1$ and the unit normal is $\mathbf{n}=(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ so $\mathbf{F}\cdot\mathbf{n}=2(x+y+z)/\sqrt{3}$. S is as a graph $z=1-x-y$ over the region $D=\{(x,y);\ x+y\le 1,\ x\ge 0,\ y\ge 0\}$ in the xy-plane. Then $dS=dxdy/\cos\gamma=dxdy/\mathbf{n}\cdot\mathbf{k}=\sqrt{3}\,dxdy$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_D 2(x+y+z)\,dxdy=\int_0^1\int_0^{1-y} 2\,dx\;dy=\int_0^1 2(1-y)\,dy=2y-y^2\Big|_0^1=1$$

(b) The curve C consists of 3 line segments that each contribute with $-1/2$ so the line integral is $-3/2$. The line segment C_1 from $(1,0,0)$ to $(0,1,0)$ is given by $\mathbf{R}(t)=(1-t)\mathbf{i}+t\mathbf{j}$, $0\le t\le 1$ with the orientation corresponding to increasing t. We have

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{R}=\int_0^1 \mathbf{F}\big(\mathbf{R}(t)\big)\cdot\mathbf{R}'(t)\,dt=\int_0^1(\mathbf{i}+t\mathbf{j}+(1-t)\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j})\,dt=\int_0^1 t-1\,dt=-\frac{1}{2}$$

2. (a) Let $S_1=\{(x,y,z);\ y=0,\ x^2/4+z^2/4\le 1\}$ and $S_2=\{(x,y,z);\ x^2/4+y+z^2/4=1,\ y\ge 0\}$ The area of S_1 is 4π. S_2 can be viewed as a graph $y=g(x,z)=1-x^2/4-z^2/4$ over the disc $D=\{(x,z);\ x^2/4+z^2/4\le 1\}$ in the xz-plane. With $G(x,y,z)=y-g(x,z)$ the unit normal is $\mathbf{n}=\nabla G/|\nabla G|=\big(-g_x\mathbf{i}-g_z\mathbf{k}+\mathbf{j}\big)/\sqrt{1+g_x^2+g_z^2}=\big(x\mathbf{i}/2+z\mathbf{k}/2+\mathbf{j}\big)/\sqrt{1+x^2/4+z^2/4}$. Now $dS=dxdz/\mathbf{n}\cdot\mathbf{j}=\sqrt{1+(x^2+z^2)/4}\,dxdz$. Introducing polar coordinates in the xz-plane:

$$\iint_{S_2} dS=\iint_D\Big(1+\frac{x^2+z^2}{4}\Big)^{1/2}dxdz=\int_0^2\int_0^{2\pi}\Big(1+\frac{r^2}{4}\Big)^{1/2}d\theta r\,dr=2\pi\frac{4}{3}\Big(1+\frac{r^2}{4}\Big)^{3/2}\Big|_0^2=2\pi\frac{4}{3}(2^{3/2}-1)$$

(b) The normal to S_1 is $\mathbf{n}=-\mathbf{j}$ and there $\mathbf{F}\cdot\mathbf{n}=y=0$ so the integral over S_1 vanishes. Since $\mathbf{F}\cdot\mathbf{n}=(x^2/2+z/2-y)/\sqrt{1+x^2/4+z^2/4}$ we obtain

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_D(x^2/2+z/2-y)dxdz=\iint_D\big(x^2/2+z/2-(1-(x^2+z^2)/4)\big)\,dxdz$$

Introducing polar coordinates in the xz-plane; $x=r\cos\theta$, $z=r\sin\theta$, $dxdz=rdrd\theta$;

$$\int_0^2\int_0^{2\pi}\Big(\frac{r^2}{4}(2\cos^2\theta+1)+\frac{r}{2}\sin\theta-1\Big)d\theta\,r\,dr=\int_0^2\int_0^{2\pi}\Big(\frac{r^2}{4}(\cos 2\theta+2)-\frac{r}{2}\sin\theta-1\Big)d\theta\,r\,dr=\ldots=0$$

3. The region R corresponds to the region $D=\{(u,v);\ 1\le u\le 2,\ 1\le v\le 2\}$.

Now $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2xy & x^2\\ -y/x^2 & 1/x\end{vmatrix}=3y$ and $\dfrac{\partial(x,y)}{\partial(u,v)}=\Big(\dfrac{\partial(u,v)}{\partial(x,y)}\Big)^{-1}=\dfrac{1}{3y}$ so

$$\iint_R x\,dxdy=\iint_D x\frac{\partial(x,y)}{\partial(u,v)}\,dudv=\int_1^2\int_1^2\frac{x}{3y}dudv=\int_1^2\int_1^2\frac{1}{3v}dudv=\frac{\ln v}{3}\Big|_1^2=\frac{\ln 2}{3}$$

4. (a) $\mathbf{R}_u=(-\sin u-v\cos u)\mathbf{i}+(\cos u-v\sin u)\mathbf{j}$ and $\mathbf{R}_v=-\sin u\mathbf{i}+\cos u\mathbf{j}+\mathbf{k}$ so

$$\mathbf{R}_u\times\mathbf{R}_v=\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin u-v\cos u & \cos u-v\sin u & 0\\ -\sin u & \cos u & 1\end{bmatrix}=(\cos u-v\sin u)\mathbf{i}+(\sin u+v\cos u)\mathbf{j}-v\mathbf{k}$$

Hence $dS=|\mathbf{R}_u\times\mathbf{R}_v|\,dudv=\sqrt{1+2v^2}\,dudv$.

(b) $$\iint_S z\,dS=\int_0^1\int_0^{2\pi}v\sqrt{1+2v^2}\,dudv=2\pi\int_0^1 v\sqrt{1+2v^2}\,dv=\pi\frac{(1+2v^2)^{3/2}}{3}\Big|_0^1=\pi\frac{3^{3/2}-1}{3}$$